

PROBLEMS OF USING PLANAR FINITE ELEMENTS (THIN PLATE THEORY) AND THEIR ENGINEERING SOLUTION IN THE ANALYSIS OF REINFORCED CONCRETE SLABS

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Abstract. Reliable performance of floor systems requires accurate evaluation of their stress–strain state. The study shows that real reinforced-concrete slabs often exhibit larger deflections than predicted, even when cracking is considered, because cracks in practice appear earlier than assumed in analysis – especially diagonal cracks in slab corners. This is mainly due to the limitations of the Kirchhoff-Love thin plate theory, which neglects transverse shear stresses τ_{yz} through the slab thickness. Excluding these stresses leads to incorrect prediction of principal stresses and crack initiation. 3D finite-element analysis of a corner-supported rectangular slab reveals significantly higher principal tensile and compressive stresses than those obtained from thin plate theory, particularly in corner zones and along free edges, resulting in earlier cracking and plastic deformations in real structures. The paper proposes an engineering method to account for transverse shear stresses τ_{yz} without full 3D modeling. The method is based on equilibrium of internal and external forces in a slab section and derives τ_{yz} distribution from thin-strip torsion with a coefficient for partial warping. All input data are obtained from thin plate theory or 2D finite elements, avoiding large 3D models. Comparison with 3D FEM shows good agreement.

Keywords: Kirchhoff-Love theory, tangential stresses, principal stresses, torsion, equilibrium conditions, finite elements, FEM modeling.

Introduction

Numerous studies show that actual reinforced concrete floor slabs have greater deflections than those obtained from calculations, even when cracking is taken into account in the calculations [1]. As shown in [1-3], this is due to the fact that in reality cracks form earlier than taken into account in the analysis software suite. This primarily concerns diagonal cracks forming in the corner regions of slabs. According to the Kirchhoff-Love plate theory [4-8], tangential stresses τ_{yz} acting across the thickness of the slab are not taken into account in the calculation of slabs. As a consequence, these stresses are not considered in the determination of principal stresses. In turn, incorrect determination of the principal stresses leads to incorrect determination of the moment of crack formation.

It is known [1; 3; 9; 10] that when a rod in the form of a thin strip is torsioned, half of the torque is absorbed by stresses τ_{xy} acting in the direction of the long side of the cross-section. The other half of the torque is absorbed by the tangential stresses τ_{yz} , which are concentrated on the vertical (side) edges of the strip cross-section. The neglect of this stress component in thin-plate theory leads to significant errors in the evaluation of shear forces near slab supports, along free edges, and in the vicinity of concentrated loads and openings [1-3].

At present, when designing floors, software packages [11] are typically used, which implement the finite element method [12]. When modelling floors, flat finite elements of thin slabs or shells are usually modelled [11; 12]. Most often, such modelling is justified. Displacements, bending moments M_x , M_y , and torsional moments M_{xy} (which are determined only from horizontal tangential stresses τ_{xy}) are determined quite accurately. However, with regard to shear forces – particularly in the regions noted above – the problem remains unresolved.

All of the above-mentioned aspects have been investigated in the studies reviewed previously. However, there is no engineering methodology for stresses that are not taken into account in the slab theory and software packages. As a rule, in such cases, it is proposed to apply bar approximation [1; 3] or to use volumetric finite elements when modelling slabs. However, the use of volumetric finite elements increases the number of unknowns in the problem by an order of magnitude. In addition, the use of volumetric finite elements in Lira-SAPR and SCAD software packages is not provided for when designing the necessary reinforcement of reinforced concrete structures.

In view of the above, the objective of this paper is to analyze the influence of tangential stresses neglected in thin-plate theory on the stress–strain state of slabs, as well as to develop practical engineering recommendations for incorporating these stresses into slab design calculations.

Materials and methods

The object of the study is floor slabs analyzed in accordance with the classical Kirchhoff-Love thin plate theory. The subject of the research is the tangential stresses acting through the slab thickness that are not taken into account in standard engineering design calculations.

The research methodology is based on the use of results obtained from classical finite element plate analysis in the form of bending and torsional moments. To evaluate the tangential stresses τ_{yz} , analogy with the torsion theory of thin-walled rods is used, according to which a portion of the torsional moment is perceived by stresses concentrated on the side edges of the slab elements.

Based on the obtained values, the principal stresses are refined, which allows for a more accurate assessment of the moment of crack formation, in particular in the support and corner regions of slabs. The proposed methodology is intended for practical engineering application and can be implemented using standard software packages without increasing the complexity of the computational models.

Main part

1. Principal stress analysis

Consider a slab that is rectangular in plan and supported at its corners. Such a structural configuration is one of the most common in residential and public buildings. The slab layout is shown in Fig. 1. For the sake of symmetry, a uniformly distributed load acting over the entire slab area is assumed.

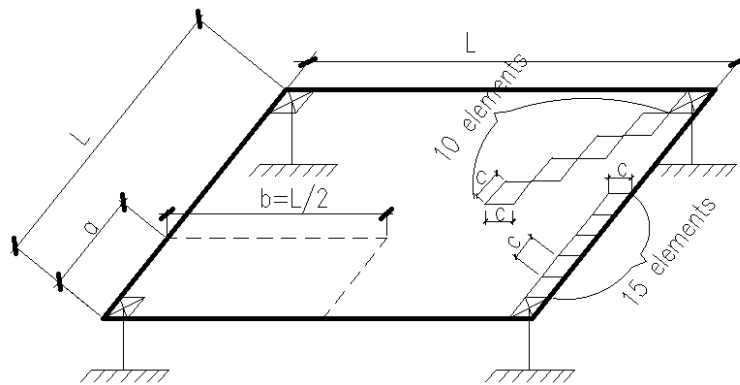


Fig. 1. Layout of a slab supported at the corners by columns

Two analysis approaches are considered. The slab is modeled using plane finite elements and volumetric finite elements. Let the slab dimensions be 3×3 m in plan with a thickness of 200 mm. For comparison purposes, two characteristic slab regions are examined (see Fig. 1). Region 1 is the corner zone along the slab diagonal, starting from the corner of the column cross-section; ten finite elements are considered in this region (the in-plane size of each finite element is $c = 5 \times 5$ cm). Region 2 is the free edge zone, beginning from the column face and extending over a length of fifteen finite elements in plan. Since a square slab is analyzed, the stress state along the remaining edges is analogous.

The analyses were performed using the LIRA-SAPR software. In the two-dimensional (plate) model, the principal stresses $\sigma_1 = \sigma_{\max}$ and $\sigma_2 = \sigma_{\min}$ were determined in each finite element. In the three-dimensional (solid) model, the slab thickness was divided into ten layers (ten finite elements through the thickness, each 2 cm thick). In this model, the principal stresses σ_{\max} and σ_{\min} were also evaluated. Table 1 presents a comparison of the principal stresses obtained from the two-dimensional and three-dimensional models.

As can be seen from Table 1, there are regions (elements in the plan) where the principal stresses in the volumetric finite element model are significantly higher than the corresponding stresses in the flat finite element plate model. This is primarily due to the neglect of the tangential stresses τ_{yz} in the plate model. Additionally, in the corner regions (Region 1), this difference is further influenced by other components of the overall stress tensor. The difference in principal maximum stresses (principal tensile

stresses) leads to cracks appearing much earlier in these areas, which increases the deformability of the slab.

Table 1

Comparison of principal stresses in a model using plane and volume finite elements

Finite element number, starting from the column	Principal stresses using two-dimensional elements, kPa		Principal stresses using three-dimensional elements, kPa		Ratio	
	σ_1	σ_2	N_1	N_2	N_1 / σ_1	N_2 / σ_2
Region 1						
1	1136.2	-490.9	2112.2	-1243.5	1.86	2.53
2	1345.9	-675.3	1797.8	-1586.9	1.33	2.35
3	1522.0	-570.8	1754.8	-1708.7	1.15	2,99
4	1627.5	-414.9	1751.2	-1737.5	1.07	4.19
5	1694.0	-249.2	1744.2	-1737.4	1.03	6.97
6	1737.4	-85.8	1732.4	-1728.2	1	20.14
7	1766.0	-71.4	1718.3	-1715.6	0.97	24.03
Region 2						
6	1692.6	-687.2	1170.6	-1185.9	0.69	1.73
7	1759.8	-601.8	1238.4	-1248.5	0.70	2.07
8	1818.4	-525.3	1305.7	-1312.0	0.72	2.50
9	1870.8	-456.5	1371.3	-1375.5	0.73	3.01
10	1918.5	-394.3	1434.9	-1437.8	0.75	3.65
11	1962.5	-338.1	1496.0	-1498.3	0.76	4.43
12	2003.5	-287.3	1554.4	-1556.2	0.78	5.42
13	2041.9	-241.4	1609.8	-1611.3	0.79	6.68
14	2077.7	-200.0	1662.1	-1663.5	0.80	8.32
15	2111.3	-162.8	1711.1	-1712.4	0.81	10.52

As can be seen from Table 1, while the principal tensile stresses are not always greater for the volumetric finite element model, the principal compressive stresses are practically always greater in the volumetric finite element model than in the planar finite element model. Considering that the concrete compression diagram exhibits a descending branch, where deformations increase significantly, the substantial increase in compressive principal stresses also contributes to greater slab deformability. Moreover, the closer to the centre of the slab span at the edge, the greater the difference. These stresses can lead to concrete spalling along the slab edges, a phenomenon frequently observed in practice. Taking this fact into account is very important in real design.

2. Engineering methodology for stress consideration τ_{yz}

It is known that neglecting the tangential stresses τ_{yz} acting through the slab thickness leads to significant errors in the assessment of crack formation and deformability of reinforced concrete floor slabs. In the vast majority of cases, designers employ flat finite elements, since the use of volumetric finite elements in practical design is problematic. Therefore, the problem arises: on the one hand, the use of flat finite elements leads to significant errors in determining tangential stresses τ_{yz} , and on the other hand, it is precisely such schemes that are predominantly used in design.

Let us consider an engineering method of accounting for tangential stresses τ_{yz} when using a scheme with flat finite elements. As noted above, the moments M_x , M_y , and M_{xy} (M_{xy} is determined only by the tangential stresses in the horizontal direction τ_{xy}) are determined without significant errors when using flat finite elements. For example, let us consider a rectangular slab supported at four corners (the most common slab layout in monolithic construction), the diagram of which is shown in Fig. 1. Let us mentally cut out the part of the slab with the lower left column in Fig. 1, along the X -axis – up to half of the slab (length b), along the Y -axis – at a distance $a < b$, and consider the equilibrium of this cut-out part along the line $y = a$, parallel to the X -axis (Fig. 2).

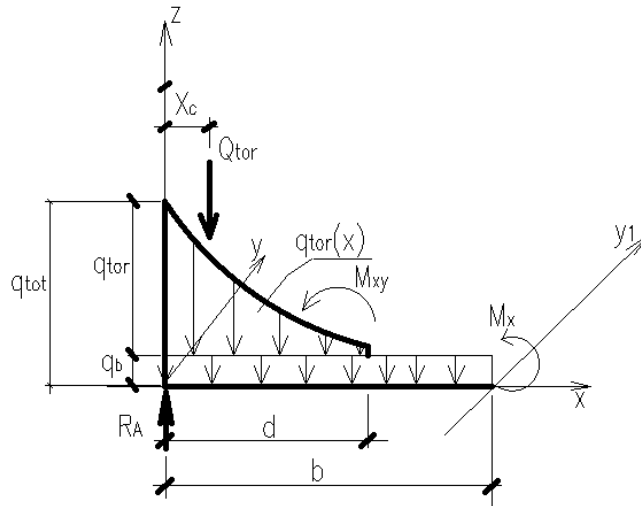


Fig. 2. Force diagram along the line $y = a$

On the side $x = b$, which is parallel to the Y -axis, the transverse force Q_x is equal to zero due to symmetry. For the same reason, the torsional moment along this line is also zero. Vertical stresses τ_{yz} and torsional moments M_{xy} from horizontal tangential stresses τ_{xy} act on the cut edge $Y = a$. The values of M_{xy} , M_x , R_A (the support reaction) are obtained from the results of structural analysis using software packages with flat finite elements. Considering that determining these values according to Kirchhoff's theory is not difficult, the values of these quantities can be obtained without necessarily using software packages.

Calculations using volumetric finite elements revealed that the distribution of vertical stresses τ_{yz} is uneven and exponentially decays from the end of the cross-section (at $X = 0$) towards the X -axis. For pure torsion of a rod in the form of a thin strip, the graph of the change in tangential stresses τ_{yz} looks like this [9; 13]. The approximation of the function of change in tangential stresses τ_{yz} for pure torsion is as follows:

$$\tau_{yz}(x) = \tau_0 \cdot e^{-\eta \cdot x} \tag{1}$$

- where τ_0 – maximum stress on the side edge of the cross section;
- η – coefficient that depends on the thickness of the rod in the form of a thin strip;
- x – coordinate along the width of the strip, starting from its vertical side surface.

In a thin strip $b/h \gg 1$ in formula (1), the coefficient η varies within small limits and is $\eta = 3/h \dots 3.5/h$, where h is the thickness of the strip, and depends little on the width of the strip.

In Fig. 2, the following designations are used: q_b – uniform internal forces per unit length (in practice, these are forces per unit length from the transverse force of the beam. We assume them to be uniformly distributed); $q_{tor} = q_{tor}(x)$ – linear forces arising from the torsion of the slab as a rod in the form of a thin strip; q_0 – maximum value of linear forces q_{tor} from torsion (to be determined); d – section across the width of the slab cross-section where the q_{tor} forces are attenuated; x_c – coordinate of the centre of gravity of the q_{tor} diagram.

The linear forces q_{tor} is the sum of the tangential stresses $\tau_{yz}(x)$ across the thickness h of the cross-section. With this in mind, let us assume that the stress in the slab is the sum of the beam stresses q_b and the torsional stresses $q_{tor}(x)$.

The value Q_{tor} in Fig. 2 is the area of the q_{tor} force diagram, which is determined from the integral from zero to d of function (1) with $q_{tor}(x)$ replacing $\tau_{yz}(x)$ and q_0 replacing τ_0 . The X_c coordinate of the centre of gravity of the $q_{tor}(x)$ curve is determined by a well-known method from mathematics for determining the centre of gravity of a curve.

The diagram in Fig. 2 contains two unknown quantities: Q_{tor} and q_b . To determine them, we will set up a system of equations – two conditions of equilibrium of the system. The first condition is that the sum of the projections of all forces onto the vertical axis Z must be zero; the second condition is that the

moments of all forces relative to axis Y_1 must be zero (see Fig. 2). Considering Fig. 2, the system of equations will look like this:

$$\left\{ \begin{array}{l} Q_{tor}(b - X_C) \cdot \xi + q_b \cdot \frac{b^2}{2} + M_{xy} \cdot \xi + M_x = R_A \cdot b - q_{ext} \frac{a \cdot b^2}{2} \\ Q_{tor} + q_b \cdot b = R_A - q_{ext} \cdot a \cdot b \end{array} \right. \quad (2)$$

In system 2, in addition to those shown in Fig. 2, the following designations are used: q_{ext} – external vertical load per unit area of the slab surface; M_{xy} – total torsional moment M_{xy} along the entire length b (parallel to the X -axis), obtained from calculations using flat finite elements; M_x – total bending moment along the length a (parallel to the Y -axis), also obtained from calculations using flat finite elements.

The coefficient ξ is a coefficient that takes into account the warping of the cross-section. With its help, in system (2), the share of torque from pure torsion (from stresses τ_{xy} and τ_{yz}) in the total torque is reduced. For a plate supported at the corners, its value is approximately $\xi = 1.05$. For calculations, you can first obtain a series of data for ξ for different cross-sections and plate sizes. Once obtained, such tabulated data can be used in calculations. If $\xi = 1$, then warping will not be taken into account. Warping can also be taken into account using the method [14].

Solving the system of equations (2), which is not difficult, gives us the values of the unknowns Q_{tor} and q_b . Thus, we have obtained simple formulas for determining the linear tangential stresses q_{tor} , which are stresses caused by torsion and are not taken into account in calculations based on the Kirchhoff-Love plate theory [4; 5; 15].

It should be noted that when comparing the data obtained using the proposed method with the data from the scheme using volumetric finite elements, the value q_{tot} should be used (see Fig. 2), because in the scheme with volumetric finite elements, the total value of tangential stresses τ_{yz} is printed. In addition, in our diagram, q_b , q_{tor} and q_{tot} are stresses (forces) per unit length of the slab, while in the volumetric finite element diagram, τ_{yz} is the stress per unit area.

Knowing the specific value of q_{tor} , it is easy to determine the distribution of tangential stresses τ_{yz} across the thickness of the plate, since q_{tor} is the area of the τ_{yz} diagram across the thickness of the plate. Considering that the distribution of stresses τ_{yz} across the thickness of the plate is parabolic [5; 14], it is not difficult to determine the maximum value $\tau_{yz,max}$ in the middle of the plate thickness, knowing the value of q_{tor} . Further, knowing the value of τ_{yz} , it is not difficult to determine the principal stresses using the formulas of strength of materials, and based on their values, to find out whether a crack is formed or not, as well as whether plastic deformations appear in the compressed concrete or not.

Extending this method to slabs supported along the contour does not present any difficulties. To do this, flat finite elements are used to calculate the reactions at the nodes of the finite elements around the perimeter of the slab. These reactions are used under the conditions of system equilibrium (2).

To verify the method, calculations were performed in the Lira-SAPR software for a 3×3 m square slab supported at the corners by columns with a uniform load of $q_{ext} = 10 \text{ kN} \cdot \text{m}^{-2}$. The calculations were performed using flat finite elements to obtain M_x , M_{xy} data along the section lines (see Figs. 1 and 2) and using volumetric finite elements to obtain real data on τ_{yz} stresses at the element edge.

Comparison of q_{tor} data at the edge of the slab obtained using the proposed method and by reducing the stresses τ_{yz} obtained from the volumetric diagram to q_{tor} showed good convergence for different values of the cross-section distance a . At the same time, the analysis of the volume model yielded a value of $\xi = 1.05$; for a slab thickness $h = 200 \text{ mm}$, $\eta = 3.07/h$ was obtained.

Thus, the developed engineering method allows adjusting the determination of tangential stresses τ_{yz} acting on the slab thickness, which are not taken into account by Kirchhoff-Love theory, using the provisions of this theory with the addition of stresses given in the proposed method.

The proposed methodology was verified through a numerical experiment using the LIRA-SAPR software package and three-dimensional solid finite elements, since the study focuses on refining the Kirchhoff-Love plate theory rather than on the physical processes in reinforced concrete. The results confirm the applicability of the method for practical structural analysis and facilitate a more accurate interpretation of future experimental studies of slabs.

Conclusions

It has been shown that, when calculating slabs using Kirchhoff's thin slab theory, tangential stresses across the slab thickness are not taken into account. This leads to significant errors in determining the stresses τ_{yz} and, as a result, the transverse forces on the free edges of the slab, near supports, openings and concentrated forces. The article presents an engineering calculation method that makes it possible to account for these stresses using plate theory, with the addition of a derived simple formula. A comparison of calculations using the developed method with calculations in the Lira-SAPR software package using volumetric finite elements showed good convergence. It is shown that the correct determination of tangential stresses allows for a more accurate determination of principal stresses and, as a result, much more accurate determination of the appearance of cracks in reinforced concrete elements. The numerical experiment confirms the applicability of the proposed methodology in practical design and its usefulness for correct interpretation of experimental studies of reinforced concrete slabs.

Research perspectives

In the future, it is planned to build a dataset for the coefficient ζ for various ratios of the plate's in-plane dimensions, thickness, and support conditions. Once obtained, such a dataset can subsequently be used in all calculations.

Author contributions

Conceptualization and methodology, T.A.; investigation, T.A., N.S., L.Ts.; writing – original draft preparation, T.A., N.S.; writing – review and editing, D.V. All authors have read and agreed to the published version of the manuscript.

References

- [1] Pereiras R. Behavior of Reinforced Concrete Floor Elements and Cross-Beam Systems with Consideration of Torsion. PhD thesis. Odesa, 2025. 248 p. (In Ukrainian)
- [2] Azizov T., Kochkarev D. Limits of Using the Theory of Plates in the Calculation of Reinforced Concrete Slabs. *Sciences of Europe*, no. 111, 2023, pp. 28-32.
- [3] Azizov T., Pereiras R., Maistrenko O. Influence of the structural support model on the stress-strain state of reinforced concrete floor slabs. *Materials Science Forum*, vol. 1100, 2023, pp. 159-165.
- [4] Love A. E. H. A Treatise on the Mathematical Theory of Elasticity. *Nature*, vol. 74, 1908, pp. 74-75.
- [5] Timoshenko S. P., Woinowsky-Krieger S. *Theory of Plates and Shells*. 2nd revised ed., McGraw-Hill Education (International ed.), released Jan 14, 2026. ISBN 9780070858206. 580 p.
- [6] Ugural A. C. *Plates and Shells: Theory and Analysis*. 4th ed. Boca Raton: CRC Press, 2017. DOI: 10.1201/9781315104621
- [7] Radwańska M., Stankiewicz A., Wosatko A., Pamin J. *Plate and Shell Structures: Selected Analytical and Finite Element Solutions*. Chichester: John Wiley & Sons, Ltd, 2016 (online book). DOI: 10.1002/9781118934531
- [8] Altenbach H., Chinchaladze N., Kienzler R., Müller W. H. (eds.) *Analysis of Shells, Plates, and Beams: A State of the Art Report*. Cham: Springer, 2020. DOI: 10.1007/978-3-030-47491-1
- [9] Timoshenko S. P., Goodier J. N. *Theory of Elasticity*. 3rd ed. New York: McGraw-Hill, 1970 (reprint: McGraw-Hill Education India, 2010; 2019). 567 p.
- [10] Steigmann D. J., Bîrsan M., Shirani M. *Lecture Notes on the Theory of Plates and Shells: Classical and Modern Developments*. Cham: Springer, 2023. DOI: 10.1007/978-3-031-25674-5
- [11] LIRA-SAPR Software Package. Calculation and Design Examples. [online] [11.02.2026] Available at: <https://www.liraland.ua/files/lira/format-pdf/> (In Ukrainian)
- [12] Zienkiewicz O. C., Taylor R. L., Govindjee S. *The Finite Element Method: Its Basis and Fundamentals*. 8th ed. Oxford: Butterworth-Heinemann (Elsevier), 2025. 752 p.
- [13] Saint-Venant A. Mémoire sur l'équilibre des corps solides, dans les limites de leur élasticité, et sur les conditions de leur résistance, quand les déplacements éprouvés par leurs points ne sont pas très-petits. *Mémoires de l'Académie des Sciences*, vol. 24, 1847, pp. 260-263.
- [14] Yaremenko O. F., Shkola Yu. A. *Bearing Capacity and Deformability of Reinforced Concrete Bar Elements in a Complex Stress State*. Odesa: Even, 2010. 134 p. (In Ukrainian)
- [15] Wight J. K., MacGregor J. G. *Reinforced Concrete: Mechanics and Design*. Boston - New York, 2012. 1157 p.