

RESEARCH INTO NON-STATIONARY HEAT TRANSFER PROCESSES FOR HYPERBOLIC TRANSFER MODELS IN GROUND HEAT EXCHANGERS INCREASING ENERGY EFFICIENCY OF HEAT PUMP HEATING SYSTEMS IN SPORTS BUILDINGS

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Abstract. The article presents a hyperbolic heat conduction equation that adequately describes non-stationary (transient) heat transfer processes in ground heat exchangers designed to improve the energy efficiency of heat pump heating systems in sports buildings and facilities. The movement of the heat transfer fluid is taken into account, and the solution to the initial-boundary value problem (Cauchy problem) for the above equation is given in a cylindrical coordinate system in three ways: a) methods of perturbation theory, which allow the analysis of thermoelastic coupled waves in a pipe, as well as using mathematical apparatus describing similar wave interactions (in linear approximation); b) by representing in analytical form the exact periodic and non-stationary solutions of the hyperbolic heat conduction equation (or the Klein-Gordon equation – linear variant) in the time domain using: 1) the Green’s function method; 2) outside the framework of Fourier series (so-called non-Fourier analysis) by analogy with the results known in the literature. The motion of a heat carrier with structural viscosity with power nonlinearity with respect to shear stress is taken into account. To account for non-isothermal heat transfer parameters, the analytical calculation of convective heat transfer is coupled with the analysis of heat transfer equation solutions (for heat transfer between two media (liquid-solid), and for each medium, heat transfer is described by its own heat transfer equation). During the modelling process, initial and boundary/edge conditions (third boundary problem) were set in accordance with the actual calculated dependencies and parameters in ground heat storage systems, as well as taking into account the main parameters (and specific operating conditions) of modern sports buildings/facilities. Within the framework of the Green’s function method, analytical solutions to the problem were obtained for the initial-boundary conditions (first and second kind).

Keywords: heat transfer processes, ground accumulator, energy efficiency, heat pump, microclimate, heating.

Introduction

To improve the energy independence and energy efficiency of heating systems at sports facilities, heat pumps with ground-source heat accumulators are often used [1-3]. A characteristic feature of such facilities is the highly dynamic thermal loads associated with competition schedules and changing operating modes [4; 5]. Classical heat conduction theory, based on Fourier’s law, assumes an infinite velocity of thermal signal propagation. However, for non-stationary processes in soils, characterized by significant relaxation times, τ_r ($3 \cdot 10^5 \dots 1.7 \cdot 10^6$ s, which ranges from 3 to 20 days), this model introduces significant errors. Transitioning to hyperbolic transfer models allows us to take into account the inertia of heat flows and accurately calculate thermoelastic stresses in heat exchanger structural components.

Currently, the vast majority of work on the hyperbolic heat equation relies on the traditional Fourier analysis, which inevitably leads to classical periodic solutions. As noted in the seminal works of D. Y. Tzou, the classical Fourier law does not account for lagging behavior, which becomes critical when modeling highly intense non-stationary processes. In real physical systems, especially during transient processes, periodic oscillations simply do not have time to form, which makes the traditional tools presented in the works of [6-8] insufficiently informative for describing the wave nature of heat. An alternative approach, developed in the works of D.Y. Tzou [9] and used in this study, correlates with modern Western concepts of non-classical heat transfer. In particular, taking into account the microscopic relaxation time of the medium allows us to move away from the parabolic model in favor of a hyperbolic one, describing “thermal shocks” and temperature jumps at the wave front. The use of “non-Fourier” analysis allows us to find exact non-periodic solutions, which are absent from standard reference books on differential equations but are critically important for describing the evolution of short pulses. This approach, which takes into account the inertia of heat flows, is most appropriate when studying the heating systems of indoor sports facilities (during their most vulnerable moments: when the system is transitioning to normal operating mode; during aggressive startup modes and emergency shutdowns; and when thermal and hydro-elastic pulses occur that could damage the piping structures).

The problem of accounting for the finite velocity of heat propagation was posed in the works of Maxwell, Cattaneo, and Lykov [6-8]. In recent decades, interest in this field has been renewed in the

context of microelectronics and laser technologies. However, the application of GSP to macroscopic systems, such as ground-based heat storage for sports facilities, has been understudied. Most existing methods [9-11] rely on numerical methods (FEM), which often blur the wavefront. This article proposes the use of analytical methods (Green's functions and non-Fourier analysis), which ensure accurate recording of temperature jumps at the interface between media.

The aim of this study is to justify and develop hyperbolic heat transfer models in ground heat exchangers using non-classical analytical tools. The work is aimed at obtaining accurate non-periodic solutions describing transient thermoelastic processes and wave fronts, which is necessary to improve the energy efficiency and operational reliability of engineering systems in sports facilities.

Materials and methods

Ground-source heat exchangers integrated with heat pumps enable efficient low-temperature heating of indoor sports facilities by utilizing the stable, low-grade thermal energy stored in the ground. These systems are particularly effective for maintaining comfortable indoor conditions in sports halls, where the required temperature typically ranges from 18 to 22 °C. They can reduce heating energy costs by 40-60% compared to conventional heating systems and significantly improve the overall energy efficiency of buildings, especially in rural areas. The performance of heat pump systems largely depends on the heat transfer processes occurring in the ground heat exchanger (thermal storage unit), from which the heat pump extracts heat and delivers it to the building's heating circuit. Standard design calculations for ground heat exchangers in heat pump applications typically rely on simplified approaches based on average heat flux per square meter or per cubic meter [1-3]. In most cases, detailed hydrodynamic and thermal analyses of the heat exchanger itself are not performed [11-14].

Analysis of the coupled thermoelasticity equation in a linear approximation. Assuming the disturbance of the medium (ground) under the action of heat sources to be relatively small, we use a linear approximation of the coupled thermoelasticity equations. In the linear approximation of coupled thermoelasticity for a homogeneous isotropic medium, the following main notations are adopted: \vec{u} is the displacement vector, K is the thermal diffusivity ($\text{m}^2 \cdot \text{s}^{-1}$), λ is the thermal conductivity ($\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}$), τ is the temperature (K), and T_0 is the initial ambient temperature (K). The linear approximation of the coupled thermoelasticity equations is valid for a homogeneous isotropic solid, a liquid, a gas (the heat exchanger structural materials), and the media filling them. Generally speaking, considering the potential nature of heat sources and neglecting vortex components that arise only at the interfaces between soil layers, the thermoelastic perturbation is represented as the sum of displacement and temperature potentials [11; 12; 14].

Let us now consider the propagation of a potential thermal disturbance in a quiescent ground structure. In the absence of internal heat sources ($Q = 0$), the governing equation describing the joint propagation of elastic and thermal waves takes the form:

$$\left(\Delta - \frac{1}{\tilde{C}_l^2} \cdot \frac{\partial^2}{\partial t^2} \right) \cdot \left(\Delta - \frac{1}{\chi} \cdot \frac{\partial}{\partial t} \right) \cdot \tau = \frac{\varepsilon}{\tilde{C}_l^2} \cdot \Delta \left(\frac{\partial^2}{\partial t^2} \tau \right), \quad (1)$$

where $\varepsilon = (\gamma - 1) \cdot \left\{ \frac{3\lambda_t + 2\mu_t}{3\lambda_t + 6\mu_t} \right\}$, $\tilde{C}_2 = \sqrt{(\lambda_s + 2\mu_s/\rho)}$ – longitudinal elastic wave velocity, $\text{m} \cdot \text{s}^{-1}$;

λ_s, μ_s – Lamé constants for an adiabatic process, $\text{N} \cdot \text{m}^{-2}$;

$\chi = \tilde{k}/(\rho \cdot \bar{C}_l)$ – thermal diffusivity coefficient, $\text{m}^2 \cdot \text{s}^{-1}$;

$\bar{C}_l = C_v \cdot (1 + \varepsilon)$ – specific heat capacity of a parallelepiped calculated for a process in which the displacement of the lateral faces is zero, and a constant pressure is maintained at the ends, $\gamma = C_p/C_v$;

C_p – specific heat capacity of the medium at constant pressure ($p = \text{const}$);

Δ – Laplace operator.

It should be noted that, $C_v < \bar{C}_l < C_p$, and for liquids (gases) $\bar{C}_l = C_p$. Equation (1) describes the joint propagation of two modes (waves) - longitudinal elastic and thermal. The measure of the mutual transformations of elastic and thermal energy is the coupling coefficient ε . For condensed media, as a rule $\varepsilon < 1$, and for gases $\varepsilon \sim 1$. However, even at $\varepsilon \sim 1$ the elastic and thermal modes can be considered independently. The conditions of connectivity and continuity of the medium are satisfied for

the frequency ranges under consideration, 10^{10} - 10^{12} Hz. Given the small value of the coupling coefficient ε , the temperature field τ can be expressed as the sum of two independent solutions:

$$\tau = \tau_a + \tau_t, \quad (2)$$

where τ_a – corresponds to a longitudinal elastic wave;
 τ_t – corresponds to the thermal (heat) wave.

The displacement vector can be represented in an analogous form as the sum of elastic and thermal components. Under the accepted assumptions and neglecting the small coupling coefficient ε , the elastic and thermal modes can be analyzed independently. The propagation of the longitudinal elastic wave occurs with the wave vector determined from the corresponding dispersion relation, while the thermal wave propagates with its own wave vector. The specific coefficients in these relations are determined from the boundary conditions of the problem.

Results and discussion

Analysis of transient elastic fields in dispersive, dissipative, deformable media (bodies) under the influence of certain wave pulses. Traditional methods for solving thermoelasticity equations in continuous media are based on the principle of separation of variables and the use of Fourier integral transforms [15-17]. This approach is effective for quasi-monochromatic waves with slowly varying amplitude-phase characteristics. However, when modeling the spatiotemporal evolution of short transient pulses in dispersive deformable media (characteristic of the startup modes of ground heat exchangers), classical Fourier analysis encounters conceptual and computational difficulties [18-20]. The scientific novelty of the proposed approach lies in the use of a “non-Fourier” analysis method, developed based on the works [9; 13; 18]. The method allows for obtaining exact analytical solutions without assumptions regarding the slowness of field changes. This is critically important for describing rapidly varying processes, where the non-stationarity of the field structure is caused by changes in the envelope over a time scale comparable to the microscopic relaxation time of the medium.

By introducing dimensionless coordinates η and time T , the system is reduced to the canonical form of the Klein-Gordon equation. The method of integral transforms is used to solve the resulting Klein-Gordon equation. Instead of cumbersome calculations, the solution is expressed in terms of modified Bessel functions of the first kind of zero and first orders (I_0, I_1). The application of standard recurrence relations and properties of transformation operators [21-23] allows the problem to be reduced to an analysis of the evolution of a wave packet. In particular, the use of asymptotic expansions of Bessel functions for large values of the argument (which corresponds to the calculation of processes over significant time intervals $\tau \gg 1$) allows the initial operator relations to be simplified to power functions of the form τ^{-n} . This makes it possible to directly estimate the decay rate of the temperature front without the need to compute the full integral series.

In [14] it is shown that for broadband short thermoelastic pulses it can be represented in the following formula:

$$\left\{ \begin{array}{l} \frac{\partial^2 \Phi_1}{\partial x_k^2} - \frac{1}{c_1^2} \frac{\partial^2 \Phi_1}{\partial t^2} = \left(\frac{3}{4a} + \varkappa \cdot \bar{\varkappa} \right) \cdot \frac{\partial \Phi_1}{\partial t}; \operatorname{Re}(\alpha_1^2) \neq 0 \\ \Phi \approx \Phi_1; \Phi_2 \rightarrow 0; \operatorname{Re}(\alpha_1^2) \rightarrow 0; \operatorname{Im}(\alpha_2^2) \approx \frac{\omega^2}{4a} \end{array} \right., \quad (3)$$

where $\operatorname{Im}(z)$ – imaginary part of a complex number;
 a – thermal conductivity coefficient, $\text{m}^2 \cdot \text{s}^{-1}$;
 ω – angular frequency, $\text{rad} \cdot \text{s}^{-1}$;
 Φ_1 – scalar potential of the thermoelastic perturbation.

Using the apparatus of special functions and integral representations [17], the solution for the temperature field is reduced to a superposition of fundamental solutions of the Cauchy problem.

Along with the solution presented in the form of integrals along the trajectories of the equation (the first one) in (3) it also describes non-sinusoidal temperature fields in the time domain using normalized variables:

$$\Phi_1 = \Phi_{10} \cdot \bar{f}, \bar{r} = t \cdot \tau_{\text{character}}^{-1}, \eta = x_k \cdot (v^* \cdot \tau_{\text{character}})^{-1}, \quad (4)$$

where $\tau_{\text{character}}$ – characteristic time for the temperature field to settle, s.

The first equation in (3) can be rewritten in dimensionless form [12]:

$$\frac{\partial^2 \bar{f}}{\partial \eta^2} - \frac{\partial^2 \bar{f}}{\partial \bar{\tau}^2} = 2 \frac{\partial \bar{f}}{\partial \bar{\tau}}. \tag{5}$$

Exact analytical solutions of the dimensionless telegraph equation (5) describing the scalar potential Φ_j variable non-periodic temperature field is represented in the form [15]:

$$\bar{f} = \sum_q \bar{a}_q \cdot \bar{f}_q, \tag{6}$$

$$\bar{f}_q = \frac{1}{2} \cdot (\theta_{q-1} + \theta_{q+1} - 2 \cdot \theta_q) = \frac{\partial \theta_q}{\partial \bar{\tau}}, \tag{7}$$

$$\theta_q = \exp(-\tau) \cdot \left(\frac{\bar{\tau} - \eta}{\bar{\tau} + \eta} \right)^{\frac{q}{2}} \cdot I_q \left(\sqrt{\bar{\tau}^2 - \eta^2} \right), \bar{\tau} \gg \eta, \tag{8}$$

where I_q – modified Bessel function; the index is determined from the limited conditions on the surface of a deformed medium $\eta = 0$.

The characteristic properties of non-separable functions (6) describing thermoelastic fields in deformable media (bodies) are briefly reduced to the following:

$$\theta_q(-\bar{\tau}, \eta) \Big|_{\bar{\tau}=\eta} = 0; (q > 0), \tag{9}$$

using the known asymptotics of functions:

$$I_q(\bar{u}) - I_q(\bar{u}) \Big|_{\bar{u} \gg 1} = \frac{\exp(-\bar{u})}{\sqrt{2\pi\bar{u}}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^n} \cdot \frac{\Gamma(q + \frac{1}{2} + n)}{\Gamma(q + \frac{1}{2} + n)}, \tag{10}$$

where Γ – gamma function (of its argument), one can find the law of decrease of the thermoelastic field \bar{f}_q in any section $\tau \gg \eta$:

$$\bar{f}_q \Big|_{\bar{\tau} \gg \eta} = -\frac{1}{2\sqrt{2\pi}} \cdot (\bar{\tau})^{-3}. \tag{11}$$

Table 1 shows the values of function $\bar{f}_q(\bar{\tau})$ (15) characterizing the law of decrease of the thermoelastic field in any section of a dispersive dissipative deformable medium (body) during the spatio-temporal evolution in it of short wave pulses arising during transient processes (start-up, operation, shutdown) of the corresponding system, using ground heat exchangers (pipelines) $\tau \gg \eta$.

Table 1

Meaning $\bar{f}_q(\bar{\tau})$ for different $\bar{\tau}$

$\bar{\tau}$	$-\bar{f}_q(\bar{\tau})$	$\bar{\tau}$	$-\bar{f}_q(\bar{\tau})$
0.1	200.0	10	$2.0 \cdot 10^{-4}$
0.2	25.0	20	$2.5 \cdot 10^{-5}$
0.3	7.4	100	$2.0 \cdot 10^{-7}$
0.5	1.6	200	$2.5 \cdot 10^{-8}$
0.8	0.4	1000	$2.0 \cdot 10^{-10}$
1.0	0.2	2000	$2.5 \cdot 10^{-11}$
30	$7.4 \cdot 10^{-6}$	50	$1.6 \cdot 10^{-6}$
300	$7.4 \cdot 10^{-9}$	500	$1.6 \cdot 10^{-9}$
3000	$7.4 \cdot 10^{-12}$	5000	$1.6 \cdot 10^{-12}$
40	$3.1 \cdot 10^{-6}$	150	$6.0 \cdot 10^{-8}$
400	$3.1 \cdot 10^{-9}$	1500	$6.0 \cdot 10^{-11}$
4000	$3.1 \cdot 10^{-12}$	15,000	$6.0 \cdot 10^{-14}$

It is also significant that non-stationary thermoelastic fields in deformable media bodies are characterized by a natural time scale $\tau_{\text{character}}$ playing a decisive role in the processes of pulsed input of related fields in continuous (deformable) media (bodies).

The analysis of non-stationary thermoelastic fields in dispersive and dissipative media demonstrates rapid attenuation of temperature along the wave front. At the same time, the efficiency of heat transfer in tubular ground heat exchangers significantly depends on the convective heat transfer between the heat transfer fluid and the inner surface of the pipe. To accurately account for this process, analytical dependencies for the local and average Nusselt number were used, obtained on the basis of the works [13; 18]. These relations allow evaluation of heat transfer intensification when using non-Newtonian fluids with structural viscosity (parameter $\chi = 1.0 \dots 1.3$) under different boundary conditions on the pipe wall. For the first-kind boundary conditions (constant wall temperature).

For boundary conditions of the first kind (specified wall temperature $t_w = \text{const}$), the local values of the Nusselt number are:

$$Nu_x^{(1)} = 1.07 \cdot (\chi \cdot Pe \cdot \frac{d}{x})^{1/3}, \quad (12)$$

where Pe – Peclet number;

d – inner diameter of the pipe, m;

χ – longitudinal coordinate along the pipe axis, m.

The local Nusselt number for a standard Newtonian fluid (when $\chi = 1.0$) and first-kind boundary conditions can be determined from the following equation:

$$Nu_x^{(0)} = 1.07 \cdot (Pe \cdot \frac{d}{x})^{1/3}, \quad (13)$$

Table 2 lists the local Nusselt numbers for structured and conventional Newtonian fluids for various pipe lengths under first-kind boundary conditions.

Table 2

Dependence of local Nusselt numbers on pipe length and parameter χ for first-kind boundary conditions ($t_w = \text{const}$)

Nu_x	χ	$Pe \cdot \frac{d}{x}$				
		10^0	10^1	10^2	10^3	10^4
$Nu_x^{(0)}$	1.0	1.0700	2.3052	4.9665	10.7000	23.0525
$Nu_x^{(1)}$	1.1	1.1045	2.3797	5.1268	11.0454	23.7966
$Nu_x^{(1)}$	1.2	1.1370	2.4497	5.2777	11.3704	24.4969
$Nu_x^{(1)}$	1.3	1.1678	2.5159	5.4204	11.6779	25.1593

An analysis of the results presented in Table 1 shows that the local Nusselt numbers for a conventional Newtonian fluid and for a structurally viscous fluid increase sharply as the number x decreases, i.e. as the flow approaches the inlet of the heat exchanger tube. Furthermore, for identical values of $Pe \cdot d/x$, the local Nusselt number increases as the structural parameter χ increases, which indicates an intensification of heat transfer if the pipe contains a structurally viscous fluid rather than a conventional (Newtonian) one.

For boundary conditions of the second kind (specified heat flux through the wall, $q_w = \text{const}$) for a structurally viscous fluid:

$$Nu_x^{(2)} = 1.29 \cdot (\chi \cdot Pe \cdot \frac{d}{x})^{1/3}, \quad (14)$$

For a Newtonian fluid ($\chi = 1.0$) under second-kind boundary conditions:

$$Nu_x^{(0)} = 1.29 \cdot (Pe \cdot \frac{d}{x})^{1/3}, \quad (15)$$

Table 3 lists the values of the local Nusselt numbers for structured and conventional Newtonian fluids for different pipe lengths under second-kind boundary conditions.

Table 3

Dependence of local Nusselt numbers on pipe length and parameter χ for second-kind boundary conditions ($q_w = \text{const}$)

Nu_x	χ	$Pe \cdot \frac{d}{x}$				
		10^0	10^1	10^2	10^3	10^4
$Nu_x^{(0)}$	1.0	1.2900	2.7791	5.9877	12.9000	27.7921
$Nu_x^{(2)}$	1.1	1.3316	2.8690	6.1809	13.3163	28.6892
$Nu_x^{(2)}$	1.2	1.3708	2.9534	6.3628	13.7082	29.5335
$Nu_x^{(2)}$	1.3	1.4079	3.0332	6.5348	14.0789	30.3321

By analyzing the results in Table 3, one can draw the same conclusions as when analyzing Table 2. Comparison of Tables 2 and 3 shows that heat transfer rates are higher under second-kind boundary conditions than under first-kind conditions for the same parameters.

The average Nusselt number along the pipe length (L) for a ground heat exchanger can be expressed in the following form for first- and second-kind boundary conditions:

$$\langle Nu^{(1)} \rangle = 1.62 \cdot (\chi \cdot Pe \cdot \frac{d}{x})^{1/3}, \tag{16}$$

$$\langle Nu^{(2)} \rangle = 1.93 \cdot (\chi \cdot Pe \cdot \frac{d}{x})^{1/3}. \tag{17}$$

Then, for an unstructured conventional Newtonian fluid, we have:

$$\langle Nu^{(1)} \rangle = 1.62 \cdot (Pe \cdot \frac{d}{x})^{1/3}, \tag{18}$$

$$\langle Nu^{(2)} \rangle = 1.93 \cdot (Pe \cdot \frac{d}{x})^{1/3}. \tag{19}$$

Table 4 presents the average Nusselt numbers for structured and conventional Newtonian fluids for different pipe lengths under first- and second-kind boundary conditions.

Table 4

Dependence of average Nusselt numbers on pipe length (L) and parameter χ for first- and second-kind boundary conditions

Nu_x	χ	$Pe \cdot \frac{d}{x}$				
		10^0	10^1	10^2	10^3	10^4
$\langle Nu_0^{(1)} \rangle$	1.0	1.6200	3.4902	7.5194	16.2000	34.9018
$\langle Nu_0^{(2)} \rangle$	1.0	1.1914	4.1582	8.9586	19.3007	41.5820
$\langle Nu^{(1)} \rangle$	1.1	1.6723	3.6028	7.7621	16.7229	36.0285
$\langle Nu^{(1)} \rangle$	1.2	1.7215	3.7089	7.9905	17.2151	37.0887
$\langle Nu^{(1)} \rangle$	1.3	1.7681	3.8092	8.2066	17.6806	38.0916
$\langle Nu^{(2)} \rangle$	1.1	1.9923	4.2923	9.2474	19.9230	42.9228
$\langle Nu^{(2)} \rangle$	1.2	2.0509	4.4186	9.5196	20.5093	44.1860
$\langle Nu^{(2)} \rangle$	1.3	2.1064	4.5381	9.7770	21.0639	45.3808

Analysis of the obtained results reveals that an increase in the structural parameter χ leads to higher values of both local and average Nusselt numbers. The highest Nusselt numbers occur under second-kind boundary conditions.

Let us now consider the Green's function method for solving hyperbolic-type heat conduction boundary value problems. The fundamental solution of the Cauchy problem for the hyperbolic equation was used to construct analytical solutions for the initial-boundary value problems of the first, second, and third kind. In essence, $\Psi_n(M)$ and α_n^2 denote the eigenfunctions and eigenvalues of the

corresponding spectral problem according to the boundary conditions of the first or second kind. Further, the semi-infinite domain $\bar{\Omega} = (x \geq l, t \geq 0)$ is considered as an illustrative example. This case corresponds to a ground heat exchanger immersed in the ground to a fixed depth. The hyperbolic heat conduction equation takes the form:

$$\frac{\partial T(x, t)}{\partial t} = a \cdot \frac{\partial^2 T}{\partial x^2} - \tau_r \cdot \frac{\partial^2 T}{\partial t^2}, (x, t) \in \Omega, \tag{20}$$

with initial conditions

$$T(x, t)|_{t=0} = T_0, \left. \frac{\partial T(x, t)}{\partial t} \right|_{t=0} = 0, x \geq l. \tag{21}$$

Let us introduce dimensionless variables of the following form:

$$\begin{aligned} Z &= \frac{(x-l)}{\sqrt{a \cdot \tau_r}}; \tau = \frac{t}{\tau_r}; Bi^* = h\sqrt{a \cdot \tau_r}; W(z, r) = \frac{T(x, t) - T_0}{T_c - T_0}, \\ \varphi_i^*(\tau) &= \frac{\varphi_i(t) - T_0}{T_c - T_0}; i = (1,3); \varphi_i^*(\tau) = \frac{\sqrt{a\tau_r} \cdot \varphi_2(t)}{\lambda(T_c - T_0)} \end{aligned} \tag{22}$$

where $T_c > T_0$ – selected unit of scale in the temperature scale.

Exact analytical solutions were derived for the semi-infinite domain corresponding to the ground heat exchanger under boundary conditions of the first, second, and third kinds.

Using the Laplace transform and the properties of the Green’s function, the analytical solution in the original domain was obtained in the form involving the Bessel function of the first kind and the Heaviside step function. This solution clearly demonstrates the wave nature of heat propagation with a finite velocity and the presence of a temperature jump at the wave front.

In the space of originals, we find:

$$\frac{W(z, p)}{\varphi_0^*} = \left\{ f(\tau - z) \exp\left(-\frac{z}{2}\right) + \frac{z}{2} \int_z^\tau f(\tau - \tau') \exp\left(-\frac{\tau'}{2}\right) \frac{I_1\left[\frac{1}{2}\sqrt{(\tau')^2 - z^2}\right]}{\sqrt{(\tau')^2 - z^2}} d\tau' \right\} \cdot \eta(\tau - z), \tag{23}$$

where I_1 – Bessel function of the first kind of its argument;

$\eta(\tau - z)$ – Heaviside function of its argument;

$$\eta(\tau - z) = \begin{cases} 1, & \tau - z > 0 \\ 0, & \tau - z < 0 \end{cases} \frac{d}{d(\tau - z)} [\eta(\tau - z)] = \delta(\tau - z).$$

A distinctive feature of the analytical solution (23) is the wave nature of the heat conduction process, which is manifested by the presence of the Heaviside step function. At any given time $\tau > 0$, the heat transfer medium is divided into two regions: an undisturbed region ($z > \tau$), where the temperature remains unchanged, and a heat trace region ($z < \tau$). This behavior is particularly important for transient processes in ground heat exchangers, such as system startup, shutdown, or reversal of the coolant flow. In these cases, at any point in the ground located at a distance z from the boundary ($z = 0$), the temperature begins to change only after the time $z = \tau$. On the propagating heat wave front (where $z = \tau$), the temperature experiences a discontinuity (jump). The magnitude of this temperature jump for all three types of boundary conditions can be determined using the delay theorem in the Laplace domain. The magnitude of the temperature jump $|\Delta|$ for all three boundary conditions can be calculated using the following formula:

$$|\Delta| = \lim p \begin{cases} P \left[\left(\frac{1}{P} \right) \cdot \exp\left(-\frac{z}{2}\right) \right] = \exp\left(-\frac{z}{2}\right) \\ \left[\left(\frac{\sqrt{1+P}}{P^{3/2}} \right) \cdot \exp\left(-\frac{z}{2}\right) \right] = \exp\left(-\frac{z}{2}\right), \\ P \left[\left(\frac{Bi^* \sqrt{1+P}}{P(Bi^* \sqrt{1+P} + \sqrt{P})} \right) \cdot \exp\left(-\frac{z}{2}\right) \right] = Bi^* \exp\left(-\frac{z}{2}\right), \end{cases} \tag{24}$$

where P – value of the dimensionless coordinate at the wavefront at the moment of dimensionless time.

These results are given for the first and second boundary value problems in coordinates (x, t) at $\varphi_0^* = 1$ in Fig. 1.

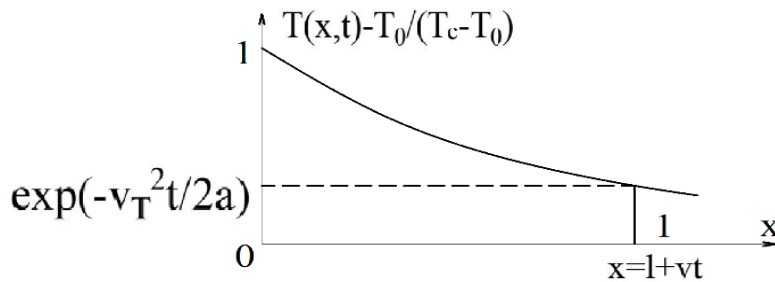


Fig. 1. Characteristic temperature profile at a given moment in time t with a discontinuity at the wave front (for the first and second boundary value problems)

For the third boundary value problem, the characteristic temperature profile at a moment in time t with a discontinuity at the wave front (for different values of the parameter Bi^* $Bi_1^* < Bi_2^* < Bi_3^*$) is shown in Figure 2.

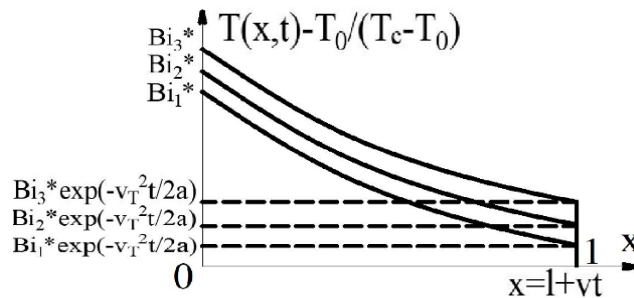


Fig. 2. Characteristic temperature profile at a time t with a discontinuity at the wave front for the third boundary value problem and different values of the parameter Bi^*

Thus, the obtained analytical solutions of the hyperbolic transport models show that a thermal shock wave propagates in a (ground) medium at a velocity of v_t . The magnitude of the temperature discontinuity $|\Delta|$ decreases rapidly (exponentially) over time.

Conclusions

1. Three models are proposed for the analysis of boundary value problems of non-stationary heat conduction for a hyperbolic equation, taking into account the relaxation time τ_r in soils from $3.0 \cdot 10^5$ to $1.7 \cdot 10^6$ s: a) a model of coupled waves in the linear approximation; b) a model of the spatio-temporal evolution of wave formations (thermoelastic type), analyzed within the framework of “non-Fourier” analysis, where in dimensionless time $\tau = 1.0$ the value of the field function f_q is 0.2 and at $\tau = 5000$ decays to $1.6 \cdot 10^{-12}$; c) a model of heat transfer in a hyperbolic medium, for which exact analytical solutions were obtained using Green’s functions and modified Bessel functions of the first kind I_0, I_1 .
2. To improve heat transfer efficiency in tubular ground heat exchangers, it is recommended to establish second-kind boundary conditions on the inner pipe surface, to use a structurally viscous heat transfer fluid with the highest possible value of χ , and to employ short heat exchanger sections (small d/L). These dependencies allow more accurate implementation of third-kind boundary conditions in the hyperbolic heat conduction model and contribute to increasing the overall energy efficiency of the heat pump system. The introduction of non-linearly structured fluids as a heat transfer medium allows the Nusselt number to be increased by a factor of 1.3 to 1.5, which means improved heat transfer efficiency under otherwise identical conditions.
3. Boundary conditions for hyperbolic transport models in integral differential forms were formed, and boundary value problems for a semi-infinite region were considered, analytical (periodic and

non-periodic) solutions were obtained, their numerical and qualitative analysis was carried out, and temperature jumps at the front of the emerging thermal wave were calculated.

4. It has been proven that analytical solutions for ground heat exchangers describe thermal shock waves, where the position of the front is strictly determined by the equality $z = \tau$. It has been established that at any given moment the environment is divided into a thermal trace zone ($z < \tau$) and an absolutely unperturbed region ($z > \tau$), where the temperature change is equal to 0. The magnitude of the temperature gap at the front decays exponentially, which allows for accurate prediction of thermal stresses in tubular structures under start-up conditions.

Author contributions

Conceptualization, Y.C.; methodology, A.M. and S.R.; software, Y.C.; validation, S.R. and P.Z.; formal analysis, Y.C. and A.M.; investigation, Y.C., A.M., S.R. and P.Z.; data curation, Y.C., A.M. and S.R.; writing – original draft preparation, Y.C.; writing – review and editing, A.M. and P.Z.; visualization, S.R. and P.Z.; project administration, A.M.; funding acquisition, P.Z. All authors have read and agreed to the published version of the manuscript.

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