QUALITATIVE ANALYSIS AND CHAOTIC DYNAMICS OF NONLINEAR THREE-DIMENSIONAL SYSTEM

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Abstract. This article presents a study of a nonlinear three-dimensional dynamic system that exhibits complex behavior. Critical points are determined analytically, and their stability is studied using the Jacobian matrix and eigenvalue analysis. Numerical methods are used to construct phase portraits and visualize attractors for different parameter values. The Lyapunov exponent spectrum is calculated to identify chaotic modes and analyze sensitivity to initial conditions. The results show that the system can have both chaotic and periodic dynamics depending on the selected parameters and confirm the effectiveness of combining analytical and numerical methods for studying nonlinear dynamic systems.

Keywords: chaotic behaviour, Lyapunov exponents, critical points, bifurcation diagrams.

Introduction

The present study is devoted to the qualitative analysis of a nonlinear three-dimensional dynamical system exhibiting complex behaviour, including chaotic dynamics. Such systems arise in various applied contexts and serve as important test models for studying nonlinear phenomena and chaotic dynamics. From the perspective of chaos theory, the considered nonlinear system can be viewed as a prototypical model for describing complex dynamics in applied fields such as economics [1; 2], where sensitivity to initial conditions plays a crucial role. The main objective of the study is to perform a rigorous investigation of the critical points of the system and to analyse their local stability properties. First, all critical points are determined analytically by solving the corresponding stationary equations. Conditions for the existence of critical points are derived depending on the system parameters. The local behaviour of solutions in the neighbourhood of each critical point is studied using linearization techniques [3]. The Jacobian matrix is constructed, and the eigenvalue spectra are analysed to classify the critical points and determine their stability [4]. To complement the analytical results, numerical methods are employed to compute the Lyapunov exponents of the system. The Lyapunov spectrum is used to detect chaotic regimes and to confirm the sensitivity of solutions to initial conditions. The presence of positive maximal Lyapunov exponents indicates chaotic dynamics and agrees with the qualitative conclusions obtained from the stability analysis [5; 6]. Furthermore, numerical simulations are carried out to construct phase portraits and visualize attractors of the system for various parameter values and initial conditions. The obtained attractors illustrate the coexistence of regular and chaotic solutions and reveal the influence of system parameters on the global dynamics.

Materials and methods

The critical points of the system are obtained algebraically by solving the corresponding equations. Numerical simulations are used to construct the graphs of the system solutions, 2D projection of the attractor on the subspace, the three-dimensional attractor, the dynamics of the Lyapunov exponents, and the bifurcation diagram with respect to the parameter $\boldsymbol{a}$. All computations and visualizations are performed using Wolfram Mathematica.

3D systems

Consider the three-dimensional system

$$\begin{cases} \dfrac{dx}{dt} = z + (y-a)x \\ \dfrac{dy}{dt} = 1 - by - dx^3 - x^2 \\ \dfrac{dz}{dt} = -x - cz \end{cases} . \tag{1}$$

The variables $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ correspond to the interest rate, investment demand, and price exponent, respectively. Parameter $\boldsymbol{a}$ describes household savings, $\boldsymbol{b}$ reflects the investment cost, and $\boldsymbol{c}$ represents the demand elasticity in commercial markets. All parameters are assumed to take positive values.

The parameters of the system are chosen as

$$a = 2;\ b = 0.1;\ c = 1;\ d = 0.1. \tag{2}$$

The corresponding initial conditions are

$$x(0) = 0.5;\ y(0) = 3; z(0) = -0.4. \tag{3}$$

To determine critical points of the system, the right side of the third equation in system (1) is first set equal to zero:

$$-x - cz = 0.$$

Then

$$z = -\frac{x}{c} \tag{4}$$

Substitution of expression (4) into the first equation of the system (1) gives

$$-\frac{x}{c} + (y - a)x = 0. \tag{5}$$

It can be factorized as

$$x\left(y - a - \frac{1}{c}\right) = 0. \tag{6}$$

Equation (6) leads to two possible cases. Let us consider the first case, where $x = 0$. Then, according to (4), $z = 0$. Substituting $x = 0$ into the second equation of the system (1) and setting the result equal to zero yields

$$1 - by = 0. \tag{7}$$

From equation (7) $y = \frac{1}{b}$, using the parameter values in (2), the first equilibrium point is: $E_1 = (0, 10, 0)$.

In the second case,

$$y - a - \frac{1}{c} = 0.$$

Then

$$y = a + \frac{1}{c}, \qquad z = -\frac{x}{c}\cdot$$

Substituting $y = a + \frac{1}{c}$ into the second equation of the system (1) and setting it to zero,

$$1 - b\left(a + \frac{1}{c}\right) - d\,x^3 - x^2 = 0. \tag{8}$$

After simplifying, this expression becomes

$$dx^3 + x^2 = 1 - b\left(a + \frac{1}{c}\right). \tag{9}$$

And finally

$$x^2(dx + 1) = 1 - b\left(a + \frac{1}{c}\right). \tag{10}$$

Taking into account the parameters in (2), the roots of equation (10) produce critical points:
$E_2 = (0.8049, 3, -0.8049)$, $E_3 = (-0.8759, 3, 0.8759)$, $E_4 = (-9.9290, 3, 9.9290)$.

After identifying all equilibrium points, their local stability is investigated by constructing the Jacobian matrix of the system. The Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} y-a & x & 1 \\ -3dx^2-2x & -b & 0 \\ -1 & 0 & -c \end{pmatrix}. \tag{11}$$

Therefore, the characteristic matrix is

$$J - \lambda \mathrm{I} = \begin{pmatrix} y-a-\lambda & x & 1 \\ -3dx^2-2x & -b-\lambda & 0 \\ -1 & 0 & -c-\lambda \end{pmatrix}. \tag{12}$$

and the corresponding characteristic equation is

$$det(J-\lambda I) = -b-\lambda+(-1-\lambda)(ab+2x^2+3dx^3-by+a\lambda+b\lambda-y\lambda+\lambda^2)=0. \tag{13}$$

Substituting the first critical point $E_1 = (0,10,0)$ into (13), the eigenvalues are obtained as

$$\lambda_1 = -0.8875, \quad \lambda_2 = -0.1, \quad \lambda_3 = 7.8875. \tag{14}$$

Since one eigenvalue is positive, the point $E_1 = (0,10,0)$ behaves as a saddle point and is unstable.

For the second critical point $E_2 = (0.8049, 3, -0.8049)$ substitution into (13) gives

$$\lambda_1 = -0.7469, \qquad \lambda_{2,3} = 0.3234 \pm 1.3431i. \tag{15}$$

Because the complex conjugate pair has a positive real part, the point E_2 behaves as a saddle-focus point and is unstable.

For the third equilibrium point $E_3 = (-0.8759, 3, 0.8759)$ the eigenvalues are

$$\lambda_1 = -0.7386, \qquad \lambda_{2,3} = 0.3193 \pm 1.3048i. \tag{16}$$

Therefore, $E_3 = (-0.8759, 3, 0.8759)$ is also a saddle-focus point and is unstable.

Finally, for the fourth critical point $E_4 = (-9.9290, 3, 9.9290)$ substitution into (13) gives

$$\lambda_1 = -9.3316, \qquad \lambda_2 = -1.0096, \qquad \lambda_3 = 10.24119549542542. \tag{17}$$

This result indicates that the critical point $E_4 = (-9.9290, 3, 9.9290)$ is a saddle point and is unstable.

To illustrate the dynamical behaviour of the system, we present the phase trajectories in Fig. 1 and 2. Fig. 3 and 4 illustrate the solutions $(x(t), y(t), z(t))$ of system (1) and the corresponding 3D attractor.

Consider $a = 2$.

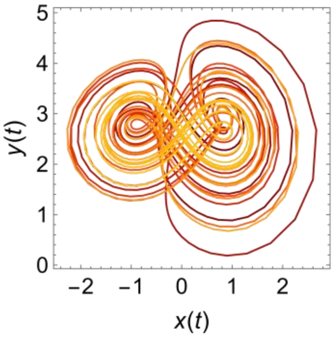

Fig. 1. **Projection of the attractor on 2D subspace $(x(t), y(t))$**

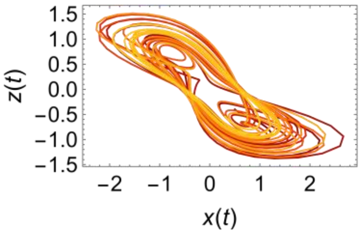

Fig. 2. **Projection of the attractor on 2D subspace $(x(t), z(t))$**

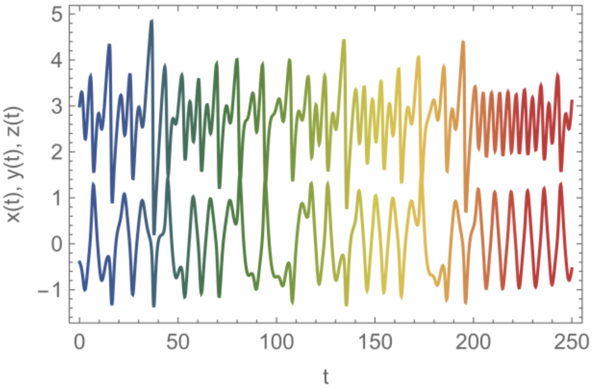

Fig. 3. **Solutions $\left(x(t), y(t), z(t)\right)$ of system (1)**

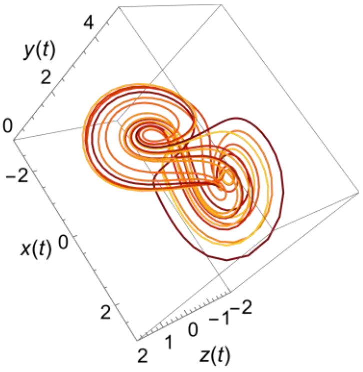

Fig. 4. **3D image of the attractor**

The calculation of the complete Lyapunov exponent spectrum is a mathematically challenging task. The computations are carried out using Wolfram Mathematica. Fig. 5 illustrates the dynamics of the Lyapunov exponents.

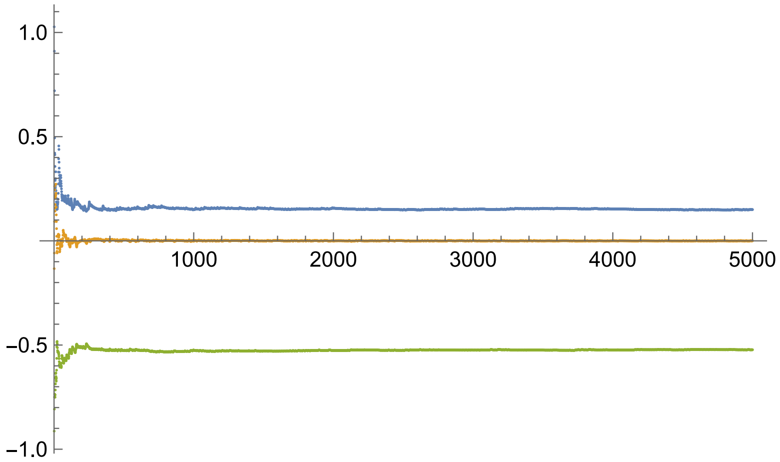

Fig. 5. **Dynamics of Lyapunov exponents $LE_1 = 0.15; LE_2 = 0;\ LE_3 = -0.5226$**

According to the Kaplan–Yorke formula

$$D_{KY} = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 2.29.$$

The calculated fractal dimension confirms the chaotic dynamics of the proposed system. Fig. 6 and 7 show the projections of the attractor onto the two-dimensional subspaces $(x(t), y(t))$ and $(x(t), z(t))$, respectively. Fig. 8 and 9 illustrate the solutions $(x(t), y(t), z(t))$ of system (1) and the corresponding 3D attractor. Fig. 10 illustrates the dynamics of the Lyapunov exponents.

Consider $a = 7$.

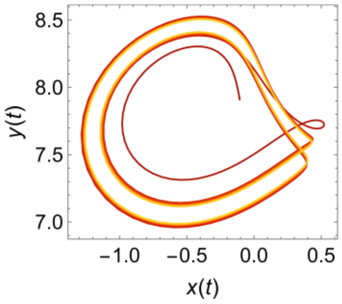

Fig. 6. **Projection of the attractor on 2D subspace $(x(t), y(t))$**

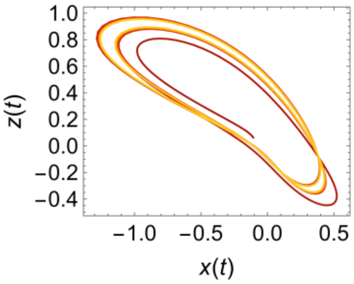

Fig. 7. **Projection of the attractor on 2D subspace $(x(t), z(t))$**

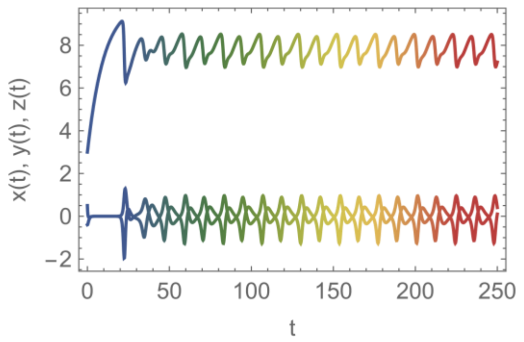

Fig. 8. **Solutions $\left(\boldsymbol{x(t), y(t), z(t)}\right)$ of system (1)**

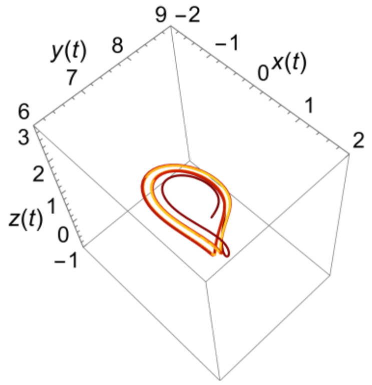

Fig. 9. **3D image of the attractor**

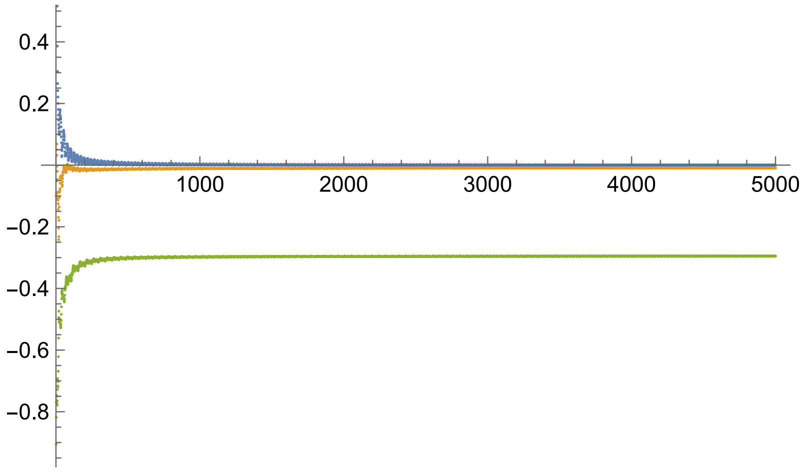

Fig. 10. $\boldsymbol{LE_1 = 0; LE_2 = -0.009;\ LE_3 = -0.295}$

According to the Kaplan–Yorke formula

$$D_{KY} = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 1.97.$$

Since the largest Lyapunov exponent is zero and the remaining exponents are negative, the system exhibits a stable periodic regime rather than chaotic dynamics [8].

To investigate changes in the system dynamics, a bifurcation diagram is constructed by varying the control parameter a, while the system parameters are chosen as follows: $a = 2$, $b = 0.1$, $c = 1$, and $d = 0.1$. Bifurcation analysis is an effective tool for identifying transitions between periodic solutions and chaotic behavior in nonlinear systems [9; 10]. The bifurcation diagram is presented in Fig. 11.

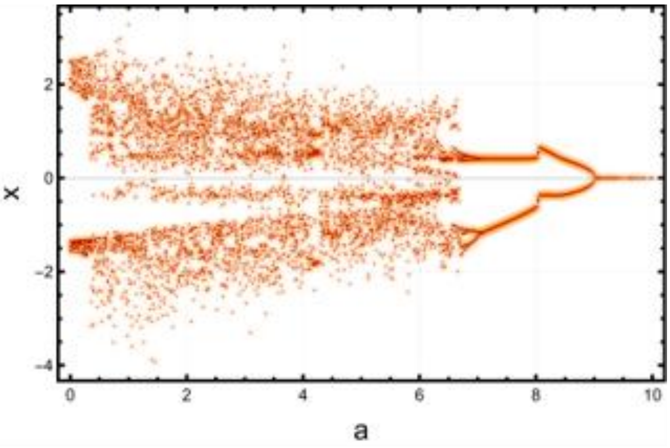

Fig. 11. **Bifurcation diagram**

Results and discussion

The analytical investigation of system (1) allowed the determination of all critical points of the system. Their local stability was studied using the Jacobian matrix and eigenvalue analysis. The obtained eigenvalues show that the critical points are unstable and correspond to saddle or saddle–focus type behaviour. This instability indicates the possibility of complex dynamics in the system.

To analyse the global behaviour of the system, numerical simulations were carried out using *Wolfram Mathematica*. Phase trajectories and projections of the attractor on two–dimensional subspaces were constructed. The obtained phase portraits demonstrate bounded but irregular trajectories, indicating the presence of a chaotic attractor. The three-dimensional attractor further illustrates the complex structure of the system dynamics. The chaotic behaviour is confirmed by the computation of the Lyapunov exponent spectrum. For certain parameter values, the maximal Lyapunov exponent is positive and confirms the chaotic regime. For another set of parameter values, the largest Lyapunov exponent becomes zero and the remaining exponents are negative, which corresponds to a stable periodic regime.

To study the influence of system parameters on the dynamics, a bifurcation diagram with respect to the control parameter was constructed. The diagram illustrates transitions between periodic and chaotic behaviour as the parameter varies. These results demonstrate that the considered nonlinear three-dimensional system can exhibit different dynamical regimes depending on the parameter values.

In addition to its economic interpretation, the considered nonlinear system can also be applied in engineering research. In particular, such dynamical systems are used to model nonlinear processes in electrical circuits, control systems, and signal processing, where complex and chaotic behaviour may arise. The obtained results on stability, bifurcations, and chaotic regimes can be useful for the analysis and design of engineering systems requiring control of nonlinear dynamics and prevention or utilization of chaos.

Conclusions

1. The critical points of the nonlinear three-dimensional system (1) were determined analytically.
2. The local stability analysis based on the Jacobian matrix and eigenvalue spectra showed that all identified critical points are unstable and correspond to saddle or saddle–focus type behaviour.
3. Numerical simulations demonstrated the existence of complex dynamical regimes, including chaotic attractors, which were visualized using phase portraits and three-dimensional attractor representations.
4. The computation of the Lyapunov exponent spectrum confirmed the presence of chaotic dynamics through a positive maximal Lyapunov exponent.
5. The bifurcation diagram illustrates transitions between periodic and chaotic behaviour depending on the control parameter, demonstrating the rich dynamical structure of the proposed system.

Acknowledgements

The work was developed within the framework of the EU ERDF-funded project “RTU Doctoral Grants for Supporting Scientific Excellence in Smart Specialization Areas” (No. 1.1.1.8/1/24/I/007) within the framework of a doctoral grant (ID 8038).

Author contributions

Conceptualization, A.L. and I.S.; methodology, A.L. and I.S.; software, A.L. and I.S.; validation, A.L. and I.S.; formal analysis, A.L. and I.S.; investigation, A.L. and I.S.; A.L. and I.S.; writing – original draft preparation, A.L. and I.S.; writing – review and editing, A.L. and I.S.; visualization, A.L. and I.S.; project administration, A.L. and I.S.; funding acquisition, A.L. All authors have read and agreed to the published version of the manuscript.

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