ENERGY APPROACH TO STUDY ON DUFFING EQUATION

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Abstract. The Duffing-type equations are considered in the search for complicated (chaotic) behavior of solutions. The most general equation is $x'' + \delta x' + f(x) = g(t)$, where $f(x)$ may be a cubic nonlinearity with different signs. The potential $F(x) = \int_0^x f(s)\,ds$ can be two or one well function. The shortened equation $x'' + f(x) = 0$ allows for mechanistic interpretation. This interpretation considers a small ball that is rolling over the graph of $F(x)$, gaining (critical) energy to escape the wells. The perturbed equation with the external force $g(t)$ may be dissipative, and it can gain and/or lose energy. If the energy level of the unperturbed equation was about the threshold allowing the ball to escape the wells, the perturbed equation may reveal irregular behavior of solutions. If the external force is dependent on parameters, the choice of parameters associated with the chaotic behavior is a problem. The article examines these situations and follows the energy changes resulting from the changes in the external force parameters. We wish to verify the following conjecture: if the energy of a homo- or heteroclinic solution in an unperturbed equation is near the critical value, allowing the small ball to escape the wells, then, for a suitable choice of the external force, chaotic behavior may occur. If the energy level is shown to be oscillatory around this critical value, then the suitable parameters in an external force can be found. After considering the equation with damping or anti-damping terms and cubic nonlinearities, the same approach is applied to the more general equation, where the nonlinearity $f(x)$ may be polynomial of higher degree. The unperturbed equation can then have multiple period annuli. The establishment of the set of parameters (usually the amplitude and the frequency of the periodic force) is a challenging problem of practical importance.

Keywords: Duffing equation, energy function, phase plane, chaotic behavior.

Introduction

The Duffing equation is an expansion of the harmonic equation

$$x'' + x = 0, \tag{1}$$

which is central in the theory of oscillations. The Duffing equation is

$$x'' + \delta x' + f(x) = 0, \tag{2}$$

where $f(x)$ – may be any nonlinearity.

The most popular is cubic nonlinearity combined with linear terms. The second term is for damping. It is real damping, if δ is positive. It is anti-damping in the opposite case. The Duffing equation (2) has multiple applications in mechanical and structural engineering, nonlinear vibrations and acoustic, electrical and electronic circuits [1], and many other fields [2-4], including biology and neuroscience. In the perturbed variant

$$x'' + \delta x' + f(x) = g(t) \tag{3}$$

the Duffing equation is one of the canonical models of chaos. The most popular choice of $g(t)$ is a periodic function. Another instance of perturbed Duffing equations may be

$$x'' + f(x) = h \cos \omega t, \tag{4}$$

where $f(x)$ is a polynomial, or

$$x'' + \delta x' + x - x^3 = h \cos \omega t, \tag{5}$$

where δ – can be positive, or negative.

The theory of the Duffing equation is rich. Basic facts and advances can be found in multiple sources, for instance, [1-16]. It is known that the unperturbed Duffing equation cannot exhibit chaos due to the Poincaré-Bendixson theory. In contrast, the perturbed Duffing equation (3) can be chaotic. This phenomenon was considered in multiple sources. If we consider equations (4) or (5), we observe that not for an arbitrary choice of parameters can chaotic behavior be detected. Therefore, the problem arises: how to know the value of parameters capable of producing chaotic behavior. Chaotic behavior reveals

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itself through various phenomena, like geometry (strange attractors), spectra (Lyapunov curves), bifurcations (Poincaré sections), and sensitivity (sensitive dependence on the initial data). Various approaches can be applied to study chaos, for instance, Poincaré (stroboscopic) sections, period-doubling cascades, Lyapunov exponents, Melnikov method. The Melnikov method is the most rigorous analytical approach for the perturbed Duffing oscillator, but it is difficult to apply in general situations.

In this article we provide another approach based on the study of energies of perturbed and non-perturbed Duffing equation. It is well-known that the energy of any particular solution of the shortened equation

$$x'' + f(x) = 0 \tag{6}$$

is constant. Therefore, the equation is conservative. Adding the damping or anti-damping term may ensure that the energy is increased or decreased in a system, which is equivalent to the equation. Including the nonlinearity makes things even more complicated. We pose the question of what happens to energy if we consider the extended and perturbed Duffing equation. Could we use calculations of energy on some solutions to reveal chaos by detecting suitable parameters? This is what we study in this article.

Materials and methods

To achieve the announced goals, we use the method of phase planes, constructing and calculating the energy function along some solutions of the unperturbed and perturbed equations. To check the chaotic behavior, the Lyapunov exponents method is applied. Computational experiments are performed with further discussions and visualizations.

Energy

The Duffing-type equations are popular and important for both theory and practice. In the minimal form $x'' + f(x) = 0$, the Duffing-type equations are integrable and contain a lot of oscillatory and non-oscillatory behaviors. But even in this form, there are many problems that can attract investigators. For instance, this equation can contain a variety of period annuli, where different forms oscillations may coexist. Adding the damping or anti-damping term makes the equation $x'' + \delta x' + f(x) = 0$, where δ can be positive or negative. Different forms of oscillation can be observed. This equation is still autonomous and does not allow for chaotic behavior of solutions. Adding the external force can make it chaotic. Even for the periodic external force of the form $g(t) = h\, cos(\omega t)$ the chaotic behavior may not appear. So, the problem of the interaction of δ, h, and ω is challenging. In search of chaotic behavior we use an energy approach. The energy function $E = x'^2 + 2\,F(x)$ is constant on solutions of the simplified equation $x'' + f(x) = 0$, but it is variable for more general cases. We use the mechanistic approach where the small ball is imagined to walk on the graph of $F(x)$. This graph contains wells and "mountains". The small ball has a constant energy on each solution in the simplest case; we fix the critical values of energy that correspond to a ball overcoming the "mountains" and is able to travel between wells. We investigate the behavior of the energy function defined on solutions of the perturbed equation and follow the changes of energy that accompany the transition to chaos. This provides the way of choosing the external forces that are able to feed the chaos. We provide two kinds of Duffing's type equations, namely, the equation

$$x'' + \delta\, x' - x + x^3 = h\, cos\, \omega t, \tag{7}$$

and

$$x'' + f(x) = h\, cos\, \omega t, \tag{8}$$

where $f(x) = -(x + 2)\cdot(x + 0.8)\cdot x\cdot(x - 1)\cdot(x - 2)$.

The formula for energy in case of the simplest equation (6) is

$$E(t) = x'^2(t) + 2\,F(x(t)), \text{ where } F(x) = \int_0^x f(s)\, ds. \tag{9}$$

Multiplying both sides of (7) by x', we obtain

$$x''x' - x\,x' + x^3\,x' = \frac{d}{dt}\left(\frac{1}{2}x'^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right). \tag{10}$$

From this and (7), we have that the energy function for (7) satisfies

$$\frac{d}{dt}\left(\frac{1}{2}x'^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right) = x'(t)\,h\,cos\,\omega t - \delta\,x'(t)^2, \tag{11}$$

which shows energy input from the forcing ($h\,cos\,\omega t$) and dissipation from the damping ($\delta\,x'$).

Way to chaos

In this subsection we show changes in energy as the equation is consequently made more complicated. We start with the equation (7).

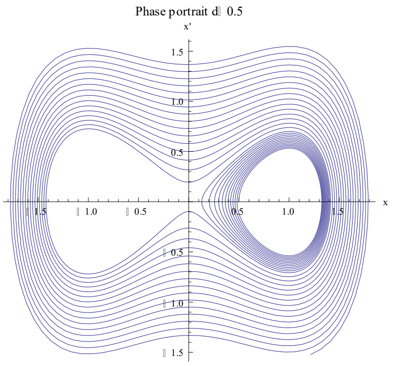


Fig. 1. **Equation:**
(10): $\boldsymbol{x''} + \delta x' – x + x^3 = 0$,
$\delta = -0.01, x(0) = 0.5, x'(0) = 0$

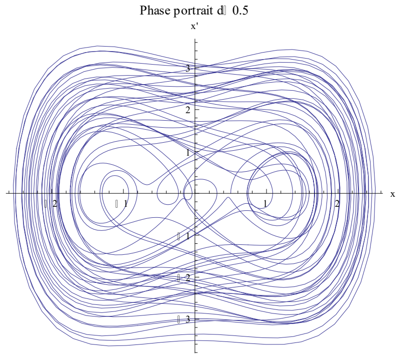


Fig. 2. **Equation:** $\boldsymbol{x''} + \delta x' – x + x^3 = h\cos\omega t$,
$\delta = -0.01, h = 1, \omega = 1, x(0) = 0.5, x'(0) = 0$

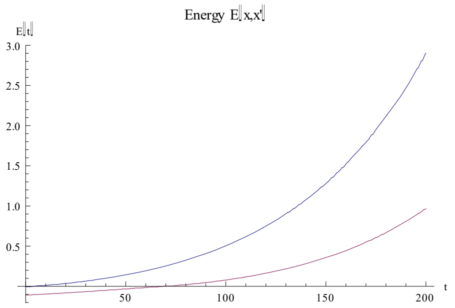


Fig. 3. **Energy of solution in Fig. 1**

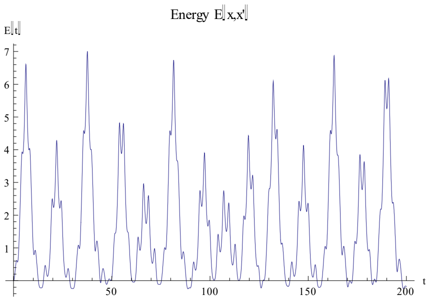


Fig. 4. **Equation**: $\boldsymbol{x''} + \delta x' – x + x^3 = h\,cos\,\omega t$,
$\delta = -0.01, h = 1, \omega = 1, x(0) = 0.5, x'(0) = 0$

The energy is injected into the system due to anti-damping. In case of damping, we have Fig. 5 and Fig 6. In both cases in Fig 2 anti-damping and Fig. 6 we see pre-chaotic behavior of solutions.

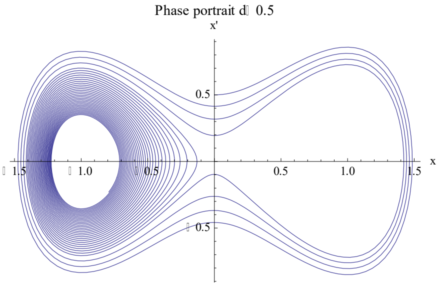


Fig. 5. **Equation:**
$x'' + \delta x' – x + x^3 = 0, \delta = + 0.01$
$x(0) = 0, x'(0) = 0.5$

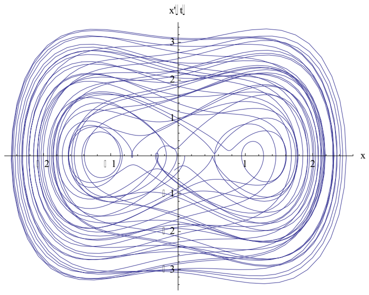


Fig. 6. **Equation:** $\boldsymbol{x''} + \delta x' – x + x^3 = h\,cos\,\omega t$
$\delta = + 0.01, h = 1, \omega = 1, x(0) = 0, x'(0) = 0.5$

The initial data are chosen so that solutions of perturbed equations start close to homoclinic solutions of shortened equation $\boldsymbol{x''} - x + x^3 = 0$. The possibility to detect chaos is then higher.

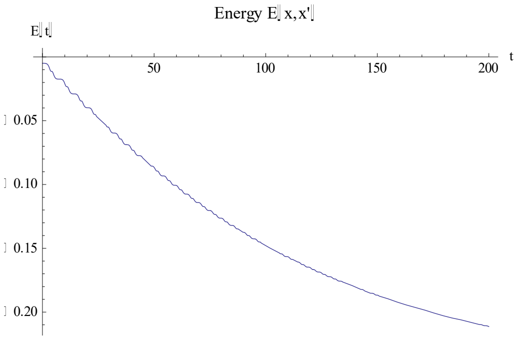


Fig. 7. **Energy of solution in Fig. 5**

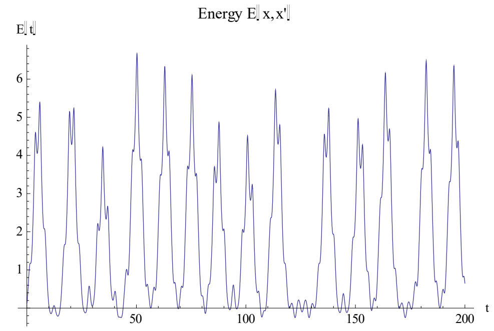


Fig. 8. **Energy of solution of equation** $x'' + \delta x' - x + x^3 = h\cos\omega t$, $\delta = +0.01, h = 1, \omega = 1, x(0) = 0, x'(0) = 0.5$

The energy in Fig. 7 decreases due to damping.

Conclusion. In both cases, damping or anti-damping, pre-chaotic behavior of solutions is observed if solutions are close to homoclinic solutions of shortened equation $x'' - x + x^3 = 0$. If perturbation includes also periodic the external force in the form $h\cos\omega t$, then pairs of parameters h, ω, favoring chaos, are those that create oscillations of energy like in Fig. 4 and Fig. 8.

Equation with several period annuli

Consider the equation

$$x'' + f(x) = 0, \text{where } f(x) = -(x+2)\cdot(x+0.8)\cdot x\cdot(x-1)\cdot(x-2). \tag{12}$$

It contains several period annuli that can be seen in Fig. 9.

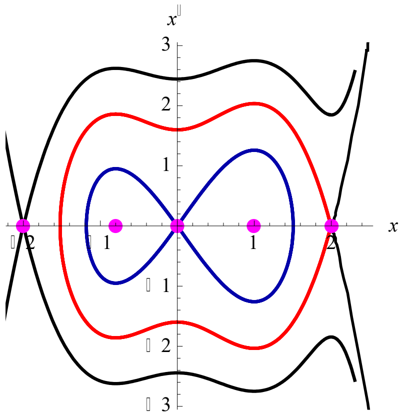


Fig. 9. **Phase portrait with period annuli** for equation (12)

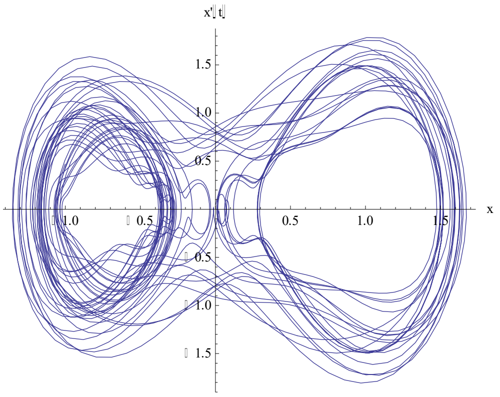
We wish to study the perturbed equation

$$x'' + f(x) = h\sin(k\,t) \tag{13}$$

and to show how knowledge of the energy behavior may help find values of h and k that are good for creating chaos. The energy function is $E(t) = x'^2(t) + 2F(x(t))$, where

$$F(x) = -\frac{1}{6}x^6 + 0.04\,x^5 + 1.2\,x^4 - \frac{4}{15}x^3 - 1.6\,x^2, \tag{14}$$

is the primitive for $f(x) = -(x+2)\cdot(x+0.8)\cdot x\cdot(x-1)\cdot(x-2)$. After perturbing the equation (13) by the periodic right side with appropriate value of k we construct a solution with the initial values slightly different from the origin in Fig. 9. The result of this experiment is depicted in Fig. 10. The corresponding energy function is in Fig. 11.



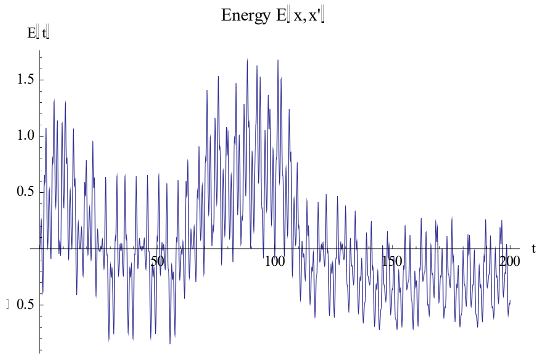


Fig.10. **Equation (13):** $h = 1, and\ k = 4.59, x(0) = 0, x'(0) = 0.05$

Fig. 11. E**nergy function for solution depicted in Fig. 10**

In Fig. 10 pre-chaotic behavior of a solution is observed. The Lyapunov curve test confirms the sensitive dependence of solutions on the initial data, that is one of main indicators of chaotic behavior.

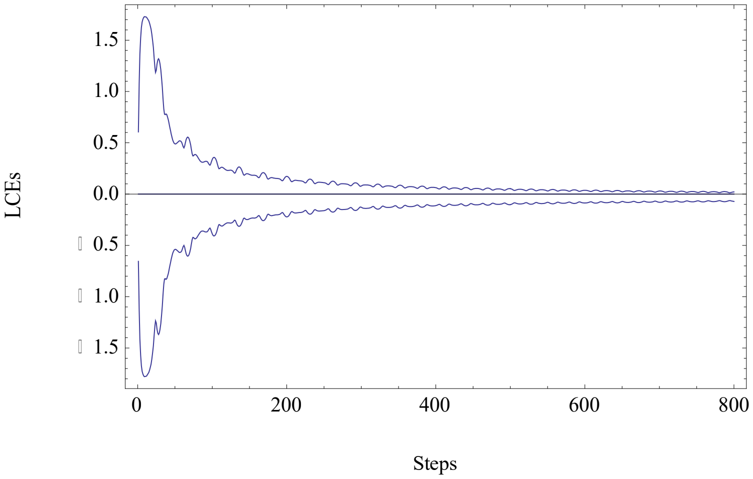


Fig. 12. **Lyapunov curves for equation (13), where** $h = 1, k = 4.59, x(0) = 0, x'(0) = 0.05$

Results, discussion and conclusions

Examples of pre-chaotic behavior of solutions in Duffing type equations of two kinds are constructed. For this, the energy function was studied numerically. The parameters were varied to get irregular behavior of the energy function. Not any perturbation can lead to chaos. To detect the suitable ones, the energy function approach can be used widely. The role of damping and anti-damping for the Duffing equation with cubic nonlinearity was revealed. No significant differences in both cases were observed. Then a polynomial nonlinearity with relatively complicated behavior was considered. It contains several period annuli. Solutions of the perturbed equation with the initial conditions close to the origin gave rise chaos. Any set of solutions arranged in a period annulus can be treated by the energy approach. This is in the perspective of further research in this direction.

Author contributions

Conceptualization, F.S.; methodology, F.S., S.A. and M.D.; formal analysis, M.D. and S.A.; investigation, F.S., S.A., and M.D.; writing – original draft preparation, F.S.; writing – review and editing, F.S. and M.D.; visualization, S.A., M.D.; project administration, M.D.; funding acquisition, M.D. All authors have read and agreed to the published version of the manuscript.

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