MODELING STABILITY OF ZIEGLER COLUMN WITH CONSIDERATION OF DRY FRICTION

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Abstract. Systems subjected to nonconservative forces, known as tracking forces, constitute an important area of research in analytical mechanics. Instability in such a system typically occurs through the onset of bifurcation (flatter or divergence), which is associated with the coalescence of natural frequencies. The modeling and analysis of nonconservative systems in mechanical systems is the primary objective of this study. A model of an inverted double pendulum loaded with a tracking force coaxial with the second arm, known as the Ziegler column, was used to conduct stability studies of the mechanical system. The Ziegler column, in its classical version, is a discrete system with concentrated-mass bodies represented by two degrees of freedom (2-DOF), originally consisting of two rigid rods connected by joints with torsion springs, loaded at the free end by a force P, the direction of which is always tangent to the last element. Equations described a simulation model with linear characteristics, and a mathematical model was constructed accordingly, which was then solved numerically. Studies of the linear model were conducted as a reference for simulations, while accounting for the nonlinear characteristics of the function describing the forces in kinematic pairs. The nonlinear model also accounted for dry friction in the column joints. It was demonstrated that while the critical flutter force was correctly identified in the linear range from a theoretical perspective, it was only in the nonlinear model that the system behavior in the supercritical state was correctly described, revealing the existence of stable limit cycles. The simulation model of the Ziegler column, in both linear and nonlinear formulations, was implemented in MSC.ADAMS software for simulating the dynamics of multi-body systems. The results of the linear model showed an exponential increase in amplitude, confirming the instability of the equilibrium point. In nonlinear modeling, due to energy dissipation through friction, a state of saturation is reached, leading to vibrations of constant amplitude (boundary cycle). Classical linear analysis allows the determination of the critical loading force P; however, it predicts an unlimited increase in amplitude once the stability threshold is exceeded, a phenomenon with no physical interpretation. In real mechanical systems, nonlinearities such as friction at structural joints limit the amplitude of vibrations.

Keywords: Ziegler column, critical load, follower force, dry friction.

Introduction

The groundbreaking works [1] and [2] form the foundation of research on the stability of nonconservative systems, focusing on the Beck-Ziegler column models. Beck [1] was the first to demonstrate that a cantilever beam subjected to a tracking force can lose dynamic stability, thereby challenging classical static theories based solely on Euler's criterion. Ziegler [2] supplemented these findings, arguing that the behavior of such systems can be correctly described only by dynamic criteria that account for the evolution of vibrations over time. These concepts were further developed in Beck's work on column geometry optimization [3] and in attempts to reconcile theory with experimental data through advanced discrete and nonlocal formulations [4], which allow for better representation of actual material properties.

A significant line of research focuses on the so-called Ziegler paradox, which holds that introducing damping into a system can paradoxically decrease the critical load and destabilize the system [5]. Recent analyses from 2025 [6] and studies on the effect of vanishing dissipation [7] shed new light on this problem by analyzing the mathematical nature of the zero-damping limit. The influence of distributed parameters on the stability of the Beck column was also the subject of analyses accounting for nonlinear hysteresis effects [8] and the overall influence of internal and external damping on bifurcation thresholds [9]. Experimental studies played a key role in verifying these phenomena, demonstrating that dry friction [10] or Coulomb friction occurring at joints [11] can also act as destabilizing factors. The results obtained for the Pflüger column [12] and the discussion of the experiments by Bigoni and Noselli [13] served as the basis for analyzing the feasibility of flutter and divergence under actual laboratory conditions, where eliminating external disturbances is extremely difficult.

A separate scientific debate concerns the physical nature of tracking forces and their implementation in engineering. Koiter [14] regarded them as purely mathematical constructs, arguing that the lack of conclusive experimental evidence precludes their existence in ideal systems. Sugiyama and his colleagues [15] held the opposite view, citing practical analogies in rocketry, pressure lines, and braking systems. However, their methodology and method of generating load were criticized in later works [16]. Elishakoff [17] summarizes this decades-long debate as still unresolved, emphasizing its philosophical and technical dimensions. Regardless of the disputes, issues of mathematical modeling remain key, including differences between systems with an infinite number of degrees of freedom and discrete models, which can lead to numerical artifacts and artificial buckling modes [18]. Contemporary research focuses on reduced-order models (ROM) [19], the influence of non-holonomic constraints [20], and advanced topological optimization of structures subjected to tracking loads [21]. Currently, flatter-type instability phenomena are increasingly widely applied in fields such as biomechanics, where models of blood flow in blood vessels are constructed, and in the engineering of metamaterials with programmable dynamic properties [22].

The main objective of this study is to model the stability of the Ziegler column, with a particular focus on the post-critical behaviour of the system and the influence of dry friction in the joints, utilizing nonlinear analysis and simulations in the MSC.ADAMS environment.

Mathematical formulation

The object of analysis is a Ziegler column, consisting of two rigid, identical rods (OA and AB) of length l and mass m each. The rods are connected to the ground via joints containing torsion springs with stiffness k. The system is subjected to a force F that maintains a direction parallel to the rod AB axis, regardless of the system configuration.

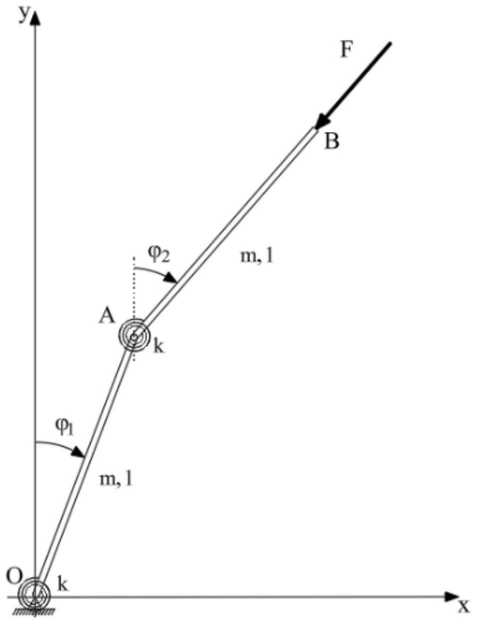

Fig. 1. **Mechanical model of the Ziegler column**

To describe the system dynamics, two generalized coordinates, $\Phi = [\varphi_1, \varphi_2]$, were adopted, where φ_1 and φ_2 denote, respectively, the angles of deviation of the rods from the vertical. The equations of motion were derived based on the second-order Lagrangian formalism:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, q_i \in \{\varphi_1, \varphi_2\}, \tag{1}$$

where $L = T - V$ (T is the kinetic energy, V is the potential energy of the system) denotes the Lagrangian of the system, and Q_i are the generalized forces arising from a nonconservative load.

Dimensionless load parameters q and time parameter τ have been introduced:

$$q = \frac{Fl}{k}, \tau = t\sqrt{\frac{k}{ml^2}} \ . \tag{2}$$

After substitution, the nonlinear equations of motion for the Ziegler column are shown below:

$$\frac{8}{3}\ddot{\varphi}_1 + \ddot{\varphi}_2\cos(\varphi_1-\varphi_2) + 2\varphi_1-\varphi_2 = q\sin(\varphi_1-\varphi_2)$$

$$\ddot{\varphi}_1\, cos(\varphi_1-\varphi_2) + \frac{2}{3}\ddot{\varphi}_2 - \dot{\varphi}_2{}^2\sin(\varphi_1-\varphi_2) - \varphi_1 + \varphi_2 = 0. \tag{3}$$

By linearizing around the equilibrium position ($\varphi_1 = \varphi_2 = 0$), the equations of motion are given below:

$$\frac{8}{3}\ddot{\varphi}_1 + \ddot{\varphi}_2 + \varphi_1(2-q) + \varphi_2(q-1) = 0$$

$$\ddot{\varphi}_1 + \frac{2}{3}\ddot{\varphi}_2 - \varphi_1 + \varphi_2 = 0. \tag{4}$$

Numerical simulation methodology

A numerical model of the Ziegler column was constructed in the MSC.ADAMS environment to validate the analytical model developed using differential variational calculus. Due to the normalized nature of the nonconservative system under consideration, the effect of gravity was completely neglected, and the motion of the components was restricted to the XY plane.

The mass and moment of inertia properties were defined in terms of the design variables. Both members were assigned an identical mass m and a mass moment of inertia about the Z-axis passing through the center of mass, equal to $I_{ZZ} = \frac{1}{2}mL^2$. The moments of inertia in the remaining axes ($I_{XX} = I_{YY}$) were assumed to be close to zero (10^{-6}) to prevent singularities in the matrix during integration. The members were connected by ideal revolute joints, into which torsion springs were introduced.

In the numerical model implemented in the MSC.ADAMS environment, the following physical system parameters were adopted: rod mass m = 1 kg, rod length l = 1 m, and joint spring stiffness k = 10 N m/rad. Since the derived equations of motion were reduced to a dimensionless form (load parameter q and time parameter t), the specific values of m, l, and k used in the simulation merely act as scaling factors. Altering them proportionally affects the actual vibration frequency and absolute force values, but it does not change the qualitative nature of the system's dynamic response or the dimensionless stability loss thresholds.

The flutter-inducing load was applied as a vector force (Action Only), oriented relative to the local coordinate system of the tip of the second member. This ensured that the load had a "tracking" character. For nonconservative systems, minimizing artificial numerical damping is critical. For this reason, the simulation was performed in Dynamics mode using the HASTIFF integrator with a reduced error step (Error = 10^{-5}).

To initiate a loss of stability, an initial disturbance in the form of a small angular velocity of the upper joint was introduced in the equilibrium state. Output data: time t, angles φ, and angular velocities $\dot{\varphi}$ were exported and then processed in the language Python for plotting nonlinear limit cycles on phase planes.

Critical force analysis of the linearized model

The stability analysis of the linearized Ziegler column model serves as a key reference point for verifying nonlinear models. The equations of motion, linearized around the equilibrium position ($\varphi_1 = \varphi_2 = 0$), can be written in matrix form:

$$M\ddot{\Phi} + K\Phi = 0, \tag{5}$$

where $\Phi = [\varphi_1, \varphi_2]^T$, while the inertia matrix M and the stiffness matrix K (taking into account the effect of the tracking force q) take the form:

$$M = \begin{bmatrix} \frac{8}{3} & 1 \\ 1 & \frac{2}{3} \end{bmatrix}, K = \begin{bmatrix} 2-q & q-1 \\ -1 & 1 \end{bmatrix}. \tag{6}$$

Assuming the solution is given by $\Phi = \Omega e^{rt}$, we arrive at the system's characteristic equation:

$$7r^4 + (54 - 15q)r^2 + 9 = 0. \tag{7}$$

From the perspective of this research, the analysis of the system loss of dynamic stability is crucial. Based on the roots of the characteristic equation, the critical force required for flutter to occur was determined to be:

$$q_{cr} = 2.54\ . \tag{8}$$

To illustrate the behavior of the linearized system in the critical state, a simulation was performed for a tracking force equal to the determined critical value. Figure 2 presents the resulting time histories and the phase plane for both generalized coordinates φ_1 (navy blue) and φ_2 (red). The time-varying vibration amplitudes and the spirals unfolding on the phase plane clearly confirm the loss of dynamic stability and the occurrence of the flutter phenomenon.

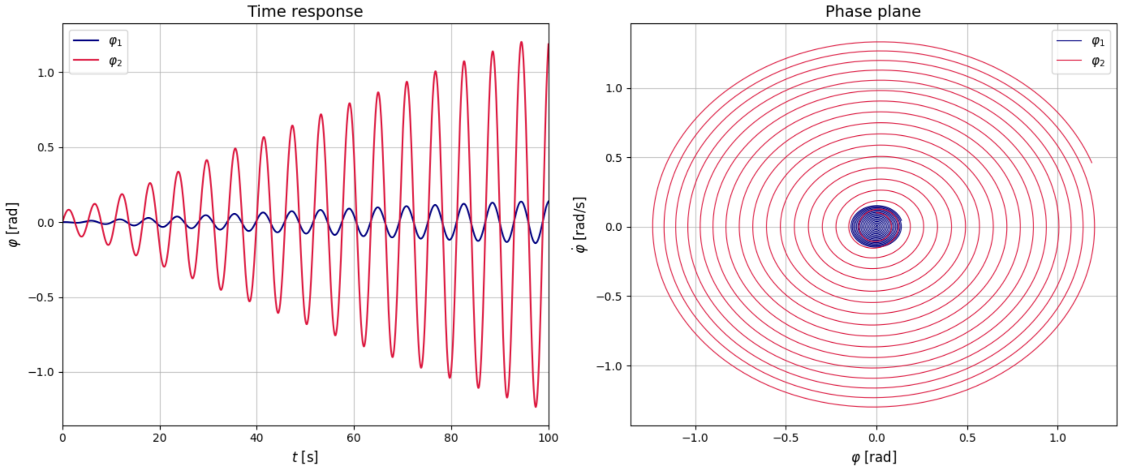

Fig. 2. **Time history and phase plane of a Ziegler column subjected to a critical load**

The determined critical flutter force ($q_{cr} = 2.54$) will serve as the primary reference point for comparisons with numerical analyses of nonlinear models in the ADAMS environment (including models that account for friction at joints).

Nonlinear analysis of dynamic instability

This chapter presents the results of numerical simulations of the dynamics of a nonlinear Ziegler column model. The studies were conducted in the MSC.ADAMS environment, omitting the effect of friction in the joints at this stage. The analysis aims to numerically verify the critical force value determined from the linearized model ($q_{cr} = 2.54$) and to investigate the system's behavior after instability.

The plots show the time histories of the rotation angles φ1 and φ2, along with their corresponding phase planes, under loading in the critical zone.

As the value of the following force was increased, a change in the system response characteristics was observed. The flutter phenomenon in the analyzed numerical model was observed for load values of $q > 2.6$. The numerically determined critical force differs slightly from the analytically calculated value $q_{cr} = 2.54$, which results directly from the inclusion of the system geometric nonlinearities, which were omitted in the analytical linear model.

Fig. 3 shows the system's behavior immediately after losing stability. The amplitude of angular vibrations initially increases over time in the form of characteristic flutter, drawing energy from the loading system. However, the phase plane demonstrates that the system enters a nonlinear flutter; the motion does not tend toward infinity but closes into a stable, complex limit cycle.

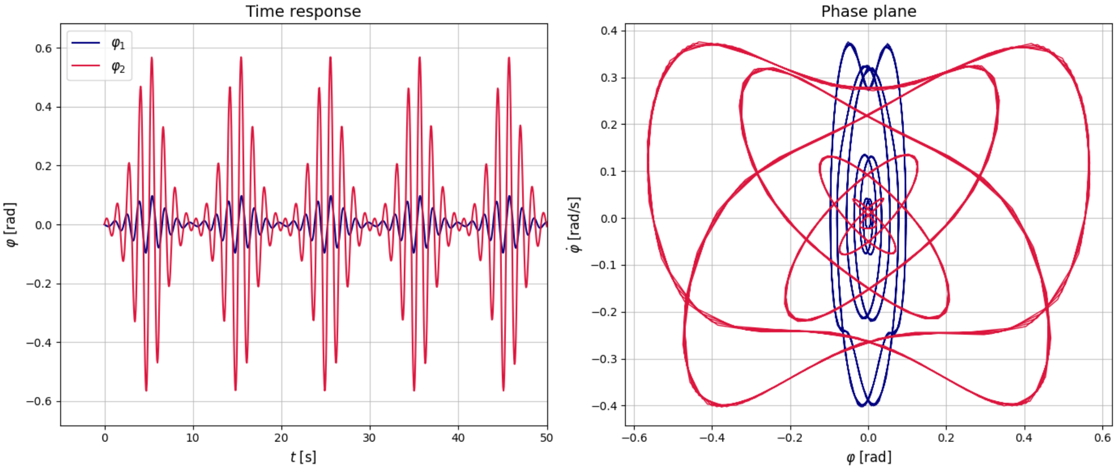

Fig. 3. **Initiation of flutter, buzzing, and the formed limit cycle for a critical load of q = 2.6**

Effect of friction in joints on system dynamics

To make the analyzed model more realistic, dry friction described by the Coulomb model was applied to the rotary joints. For the simulations in MSC.ADAMS, a velocity-dependent dry friction model was introduced to distinguish between static and kinetic friction. In the simulation, a static friction coefficient of $\mu_s = 0.12$ and a kinetic friction coefficient of $\mu_k = 0.1$ were assumed. The application of a falling friction characteristic (the difference between the static and kinetic values) is the necessary mechanism for generating the observed stick-slip self-excited vibrations, thereby ensuring the phenomenon is modelled accurately and realistically. The introduction of an energy-dissipation mechanism into the nonconservative system enabled the observation of fundamental changes in the vibration topology within the critical zone.

For a load $q = 2.6$, located just above the classical stability threshold, friction completely alters the nature of the forming attractor (Fig. 4). In contrast to the clean, precise lines observed in the frictionless model, the phase plane becomes heavily "blurred" and densely filled with trajectories.

This is a visual representation of the phenomenon of quasi-periodic oscillations caused by the stick-slip effect (alternating sticking and slipping at the joint). Nonlinear static-kinetic friction causes the system to dissipate different amounts of energy in each successive cycle, preventing the trajectory from closing perfectly and leading to a highly complex, deterministically chaotic response.

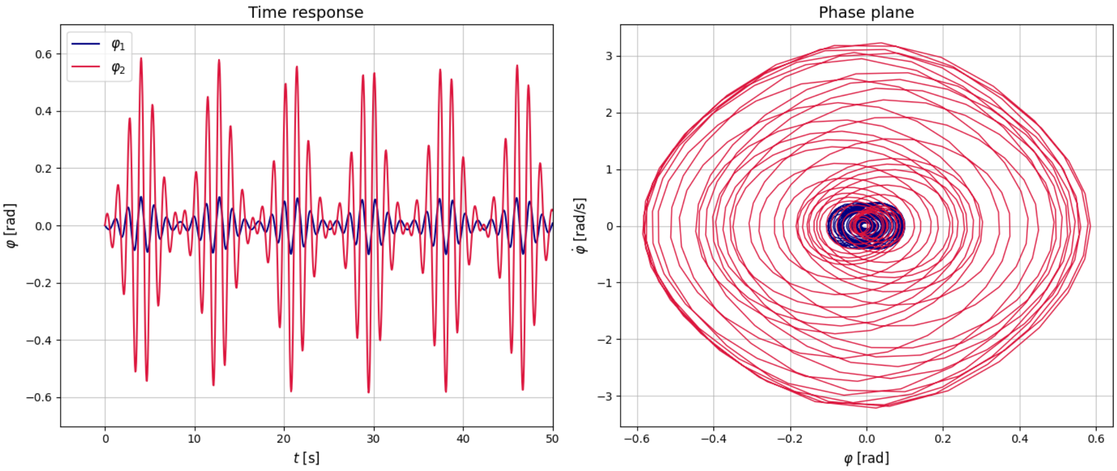

Fig. 4. **Quasi-periodic motion and the stick-slip effect on the phase plane for q = 2.6 in presence of friction**

Conclusions

A numerical analysis of the nonlinear Ziegler column model performed in the MSC.ADAMS environment, supplemented by custom Python phase-space analysis scripts, enabled a detailed investigation of the dynamics of this classic nonconservative system. The frictionless numerical model confirmed with high accuracy the moment of instability determined analytically from the linearized model. The onset of flutter is observed at a normalized load of $q = 2.6$ (compared to the analytical value $q_{cr} = 2.54$). This slight discrepancy stems directly from the model in the MSC.ADAMS environment accounts for geometric nonlinearities, which are neglected in classical Lagrange equations. It has been shown that, in a nonlinear framework, the loss of dynamic stability does not lead to an unbounded increase in vibration amplitude. Upon exceeding the critical force, the system enters a state of steady self-excited vibrations, forming boundary cycles on the phase plane.

The introduction of physically realistic dry friction into the column joints demonstrated that energy-dissipative mechanisms have a crucial influence on the behavior of nonconservative systems following loss of stability. Analysis of phase images revealed that friction fundamentally alters the system's vibration topology. Near the stability threshold, a strong stick-slip phenomenon was observed, leading to dense, quasi-periodic trajectories and destroying the regularity observed in frictionless models. This means that omitting nonlinear friction models in classical flutter analyses significantly simplifies the system's actual, deterministically chaotic operation.

Author contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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